Options Réelles et Options Exotiques, une Approche Probabiliste

Laurent Gauthier

UNIVERSITE PARIS I PANTHEON-SORBONNE

U.F.R. MATHEMATIQUES ET INFORMATIQUE

OPTIONS REELLES ET OPTIONS EXOTIQUES, UNE APPROCHE PROBABILISTE

THESE
Pour le Doctorat de l'Université Paris I en Mathématiques
par

Laurent GAUTHIER

Presentée le 27 Novembre 2002

Directeurs de recherche :

M. Marc CHESNEY, Professeur à HEC
M. Elyes JOUINI, Professeur à l'Université Paris IX et à l'ENSAE

Membres du Jury :

Mme. Monique JEANBLANC-PICQUE, Professeur à l'Université d'Evry, rapporteur
M. François QUITTARD-PINON, Professeur à l'Université Lyon I, rapporteur
M. Nizar TOUZI, Professeur à l'Université Paris I
M. Marc YOR, Professeur à l'Université Paris VI

REMERCIEMENTS

Cette thèse a été entamée en 1996 et sa rédaction s'est étalée sur environ cinq ans: 2 ans d'écriture, et 3 ans relecture et corrections. Je tiens à remercier mon directeur de thèse Marc Chesney qui, tout ce temps, m'a encadré et aiguilloné. C'est grâce à son intérêt pour les options réelles que j'ai pu découvrir ce domaine depuis 1994. Marc Chesney m'a très judicieusement conseillé, soutenu et aidé ; je lui en suis très reconnaissant.

J'ai beaucoup apprécié qu'Elyès Jouini accepte de co-diriger ma thèse et m'accueille à la Sorbonne en 1997, après que j'ai suivi quelques cours du DEA de Modélisation Mathématique de Paris-I. Elyès Jouini m'a inspiré dans mon approche des coûts de transaction, et je lui suis aussi reconnaissant de m'avoir aidé à orienter mes recherches, tout en me laissant une grande liberté.

Je tiens à remercier très chaleureusement Monique Jeanblanc-Picqué et François Quittard-Pinon, mes rapporteurs. Monique Jeanblanc-Picqué a décortiqué cette thèse, et sans elle de nombreux résultats dans ce travail seraient soit faux, soit démontrés de manière un peu trop heuristique. Parmi mes agréables souvenirs d'étudiant, le cours de calcul stochastique de Monique Jeanblanc-Picqué au doctorat HEC occupe une place particulière.

François Quittard-Pinon a accepté d'être un rapporteur, et je lui sais gré de son temps. J'espère sincèrement que cette lecture l'a intéressé et en valait la peine.

Je remercie également Marc Yor de faire partie de mon jury de thèse. Je le remercie aussi et surtout pour l'enseignement qu'il m'a dispensé en DEA. C'est à lui que je dois mes quelques connaissances sur l'étude fine du mouvement brownien.

Nizar Touzi a eu la gentillesse d'accepter de participer à mon jury de thèse, et je lui en suis reconnaissant. Les aspects mathématiques et les applications financières de cette recherche, je l'espère, l'intéresseront.

Je n'oublie pas Erwan Morellec, mon collègue doctorant et co-auteur, avec qui j'ai toujours eu des discussions intéressantes et fructueuses.

Finalement, les arbitres des revues auxquelles j'ai soumis differente parties de cette thèse ont réellement contribué à la qualité de mon travail, en relevant de nombreuses erreurs ou bien tout simplement en remettant en question ma première approche.

OPTIONS REELLES ET OPTIONS EXOTIQUES, UNE APPROCHE PROBABILISTE

Cet ouvrage se concentre sur la valorisation et la couverture d'options financières non traitées sur les marchés, les options réelles, qui servent à évaluer des décisions optimales d'investissement en capital pour des entreprises. L'existence pour une entreprise d'un projet d'investissement s'apparente en effet à la possession d'une option financière: l'entreprise possède l'option d'attendre le moment le plus favorable pour lancer son projet. Pour valoriser l'intérêt économique d'un projet, il convient alors de calculer la valeur de l'option d'investir. L'objectif de cette thèse est de montrer comment la théorie des options réelles peut bénéficier des apports des méthodes habituellement employées pour les options exotiques.

A la différence de l'approche classique dans le domaine des options réelles, qui privilégie l'utilisation de techniques d'équations différentielles, nous proposons dans cette thèse d'évaluer des projets d'investissement en applicant des méthodes très probabilistes. Cette distinction de méthode permet non seulement de généraliser l'approche classique du problème, mais encore d'obtenir des résultats analytiques dans des situations ou une technique d'équation différentielle ne permettrait pas de résoudre le problème. Egalement, c'est un pont jeté entre la recherche académique en finance d'entreprise et la floraison de nouveaux résultats sur les options exotiques, très souvent obtenus par des approches probabilistes.

Dans cette thèse, nous abordons spécifiquement des problèmes

- de valorisation de projets d'investissement sous certaines contraintes particulières : lorsqu'il existe un délai incompressible entre la prise de décision et sa mise en oeuvre, lorsqu'il existe une compétition entre deux acteurs économiques de caractéristiques différentes, et lorsque l'information sur le marché de l'entreprise est imparfaite.
- de couverture de ces projets d'investissement : comment couvrir des options réelles qui sont un peu complexes de la manière la plus efficace lorsqu'il existe des coûts de transaction sur les actifs financiers, et comment une nouvelle classe de produits dérivés qui s'apparentent aux options barrières permet de couvrir le risque lié à l'exercice des options réelles.
- de décision optimale d'investissement lorsque l'on peut manipuler le marché : un agent économique qui possède une information privilégiée sur la valeur d'une entreprise peut intervenir sur le marché afin de l'utiliser, et par la même occasion influencer la valeur des titres émis par l'entreprise. Quelle est sa stratégie optimale ?

Les outils mathématiques utilisés sont surtout probabilistes, essentiellement la théorie des excursions, les temps locaux et le contrôle stochastique. Le principal souci est l'obtention de résultats analytiques, au détriment du développement de méthodes numériques.

This work focuses on valuing and hedging financial options that are not traded, called real options, and that are used to assess corporations' optimal capital investment decisions. For a company, the existence of an investment project is similar to owning a financial option: the company possesses the option to wait for the most favorable time to launch its project. Assessing the economic attractiveness of a project therefore requires to value this option. Our objective is to show how real option theory can benefit from exotic options methods.

Unlike the classical approach in real options, which favors using differential equations techniques, we propose to value investment projects with probabilistic methods. This distinction allows not only to generalize the approach, but also to obtain analytical results in cases when a differential equation approach would not prove tractable. Also, it relates a corporate finance research domain with the very flourishing field of exotic options, a field where most results are obtained through probabilistic tools.

Specifically, we tackle

- the valuation of investment projects when there is a delay between the decision to invest and its actual implementation, when there is a competition between two economic agents with different delay-related constraints, and when the information available to the firm is noisy,
- the hedging of investment projects: how to hedge real options in the most efficient manner when there are transaction costs on the underlying assets, and how a new class of derivatives products that are related to barrier options allow to hedge the risk related to exercising real options
- the optimal investment decision of an agent who has an impact on the market. An economic agent possessing privileged information on a company can trade its stock with profit, while pushing the market price towards the price that reflects the information. What is the agent's optimal strategy?

The mathematical tools used are essentially probability, the theory of excursions, local times, and stochastic control. We are especially interested in obtaining analytical results, rather than in finer modeling or developing numerical methods.

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[1]The original version of this chapter was co-written with Erwan Morellec and is to appear in *Real Options and Investment under Uncertainty*, Eds. E. Schwartz and L. Trigeorgis, MIT Press.

[2]The part of this chapter that focuses on the alternative proofs of the Theorem of Chesney, Jeanblanc and Yor is to appear in the *Advances in Applied Probability*, under the title "Parisian Options: A Simplified Approach without Excursions"

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[3] A different version of this chapter has also been co-written with Marc Chesney, using a more analytical approach. The technical part of this chapter that focuses on the lenth and height of excursions is to appear in the *Journal of Applied Probability* in December 2002, under the title "Excursion Length and Height and Application to Finance".

[4] A version of this chapter, co-written with Erwan Morellec, has been published under the title "Noisy Information and Investment Decision: a Note", *Finance* (PUF) 1999, 20(2).

[5] A shorter version of this chapter has been published under the title "A Transaction Cost Convergence Result for General Hedging Strategies" in *Stochastic Models* (Dekker), 17(3), pp. 313-339 (2001).

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[6] This chapter has been published in the J*ournal of Applied Mathematics and Decision Sciences* (Erlbaum), 6(1), pp. 51-70 (2002), under the title "Hedging Entry and Exit Decisions: Activating and Deactivating Barrier Options".

[7] A shorter version of this chapter is to appear in the *International Journal of Applied and Theoretical Finance* under the title "Informed Opportunistic Trading and Price Optimal Control".

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Chapter 1 INTRODUCTION

The study of probability has been historically linked to decision problems. Pascal's motivation in 1654 when he wrote his "Adresse à l'Académie Parisienne" was to assess the fairest repartition of the bets if a random game was interrupted. Probability at its birth was closely associated with financial and economic issues. In 1900 Bachelier in his "Théorie de la Spéculation" provided a basis for Markovian diffusion processes, that were to be rediscovered in 1933 by Kolmogorov. Finally, Black, Merton and Scholes published in 1973 a series of fundamental papers where they valued financial products by calculating the expectation of a function of a Brownian Motion.

Since then, financial mathematics have become a flourishing branch of probability and mathematical analysis. Black, Merton and Scholes's result allowed for the development of new financial markets, inducing a significant need for research in this field. Throughout the past 25 years, considerable resources were allocated to adapting original models to new problems. It clearly appears that a symbiotic evolution took place: markets develop and mature thanks in part to new models and technical paradigms, which pushes financiers towards innovating and creating new markets so that they can maintain profitability in the business, and these new markets require new models.

A large majority of mathematical finance research has been carried out in the field of financial derivatives. These products give their owners future income streams that are a function (specified in advance) of other products' value. These other financial products are called the underlying of the derivative. From a mathematical perspective, the value of an underlying is modeled as a diffusion process, and the value of the derivative is often the expectation, under a certain probability, of a functional of this underlying process. Since 1973, the thorough study of the Brownian Motion has allowed the valuation of more and more complex derivatives, such as "barrier options" that are related to hitting times, or "mean options" related to exponential integrals of the Brownian Motion. Besides, accounting for realistic market-related constraints required the intensive use of stochastic control, in particular of the notions of viscosity solution or optimal stopping. Market "non-completion" problems (the fact there always exist a risk against which one cannot be insured) have induced significant research on the existence of equivalent probabilities, or on projections onto semi-martingale spaces.

Financial derivatives, which are traded on many markets with extremely high volumes, can also be used to model micro-economic decisions. There is a simple analogy: a company that wishes to invest in a project possesses the option to wait for better conditions to implement this investment. This option is in fact very much like a financial derivative, and its underlying consists of the economic variables that will condition the future value of the project (such as the market share, the value of the products sold or bought or the intensity of demand). This option is a so-called "real option" and its valuation is similar to that of financial derivatives. The theory of real options was developed through the 1980s, benefiting from the wide success of financial derivatives. Real options have been mentioned in Business Week[1]. For managers, real options appear as superior to the traditional method of discounting future cash flows so as to decide whether to invest or not.

[1] Coy, Peter, "Exploiting Uncertainty: The 'Real Options' Revolution in Decision-Making," *Business Week*, June 7, 1999, pp. 118-124.

As written in Business Week,

> "[By] boiling down all the possibilities for the future into a single scenario, NPV [Net Present Value] doesn't account for the ability of executives to react to new circumstances - for instance, spend a little up front, see how things develop, then either cancel or go full speed ahead"

However, as much as the field of probability benefited from the new problems posed by financial derivatives, the mathematical methods used to study real options were not demanding in terms of mathematical research. Real options being from the start more related to corporate finance than market finance, most of the research in their field has been performed on a more economic tradition. From a technical perspective, it translates into a generalized use of differential equations, rather than pure probabilistic tools. This work focuses on the frontier between exotic options and the probabilistic methods associated to them on one side, and real options on the other side. Unlike financial derivatives that are traded on various markets, there is no limit to the complexity with which the decision to invest, or the constraints under which this decision is made, can be modeled. Approaching the problem from a probabilistic angle allows a greater freedom in modeling these constraints, as we will see.

Our goal is to show how the analysis of real options can benefit from concepts and methods that are usually employed for exotic options-related problems. So as to exploit the relationships between real options, probability, and exotic options, we will articulate our work as follows. The first part of this thesis focuses on the valuation of investment projects under different constraints by the use of mathematical techniques usually applied to exotic derivatives. This part comprises Chapters 2 through 5. The second part focuses on the hedging of the risks related to investment decisions using financial options-related methods, and comprises Chapters 6 to 8.

The second chapter introduces the technical aspects of basic real option analysis from a probabilistic perspective. Then we proceed and study the effect, on the value of an investment project, of the delay existing between the decision to invest and its actual implementation. Numerous investment decisions are characterized by significant implementation delays. These delays can be linked to the search for an investment project, to the length of the decision process itself within the company, or to the time necessary to gather funding. We show that these implementation delays have an important impact on optimal investment decision rules as well as on the investment projects value. We show in particular that this loss of value can become negligible if the firm has the possibility to abandon the project during its implementation. From a technical perspective, we calculate the generating function of several stopping times related to the Brownian Motion. For example, we need to consider the first instant when a Brownian Motion spends continuously more than a given amount of time (the delay) above a barrier (the threshold that triggers the investment decision). We derive the value of this option for various exercise policies corresponding to different levels of freedom with respect to the abandonment of the project and analyze its effects on the investment policy of the firm. The mathematical results used in this Chapter are not new, but our application to real options and delay modeling had not been derived before.

The third chapter discusses the mathematical methods that can be used to study delay-related investment decision constraints. The probabilistic approach we propose allows an unequaled flexibility in specifying the various constraints faced by investing firms or managers. Some results, as we will see, cannot be reached using conventional analytical approaches. This chapter presents no fundamentally new mathematical result, but proposes a varied range of approaches to the computations involved by stopping times related the Brownian Motion and its excursions. Using mainly excursion theory, we show how to derive the generating function of some stopping times related to the modeling of delays in investment decisions. We show several proofs of the main result we used in Chapter 2 on the first instant the Brownian Motion spends consecutive time above a barrier.

The fourth chapter focuses on the competition situation between two firms interested in the same investment project, when they have different kind of constraints. The real option approach fits very well the monopolist case. However, in a competitive case, the strategic behaviors of different firms is more complex to account for. Some authors have already tackled this problem and studied the preemption of investment projects in a real option framework. The aim of this chapter is also to consider this question, but in the case where the competitors have different constraints in terms of investment delay and flexibility: one firm is large, the other is small. In our setting, the large firm suffers a delay in its investment decisions, whereas the smaller firm's decisions are instantaneously implemented. Calculating the value of the investment project, depending on the level of information available to each firm on its competitor, requires to study the first instant when a Brownian Motion spends more than a given amount of time above a certain level (that would model one firm's investment, accounting for delay) or hits another higher level (that would represent the other firm's decision). We derive the generating function of this stopping time and of other functionals by using excursion theory. To our knowledge, this result is new.

The fifth chapter studies the investment decision when the information on the underlying project is "noisy". When we look at empirical evidence, the option premium detected in those models seems to have a great statistical significance. However, most tests find that the option premium generated by the data is generally spread over and under the value generated by the models. In competitive markets the winner's curse can account for the undervaluation of the real option models. Another reason for this undervaluation may be that many models developed so far are very simple and do not account for the investment projects 'embedded options such as the option to expand the investment size or the option to abandon the project. We show in this paper that the noise in the information available to the investor can account for their overvaluation. The revenues associated with the exploitation of petroleum leases or mines is typically noisy as the extraction rate or the amount of reserves are subject to large forecasting errors. This chapter examines the effects of noise on investment decisions. We use the computation of first passage times to derive closed-form formulas relating the value of the investment opportunity to the noisy decision variable. Our setting for the description of the noise is simple but brings out the generality of the idea. We show that the value of real options, when we account for the noise, is lower than the option value computed in the perfect forecast case. For reasonable parameter values, our model can generate values for the investment opportunities very close to that observed in reality.

The sixth chapter considers the hedging of derivatives, whether they are financial derivatives or real options, using other financial derivatives. As there are transaction costs, there is a balance between the frequency of transactions and the quality of a financial product as a good hedge. For many firms, especially in the mining, oil, or commodities industries, the value of investment projects can be determined with real option theory. In these cases, an important argument underlying the valuation of projects is that the business risk, being linked to a traded product, can be hedged. Real options in that case can be considered as equivalent to complex options written on a commodity. In this paper, we address the issue of hedging such options, and more generally hedging complex options, using an optimal combination of underlying product and other derivatives written on it. The optimal hedging basket should minimize transaction costs, which can be important for derivatives, as well as for the underlying. There is a trade-off between hedging with a derivative that replicates locally well the real option but with a high transaction cost, or with the underlying at a lower transaction cost but a higher frequency of rehedging. This chapter generalizes the result of Leland (1985) to hedging strategies that use not only the underlying but all kinds of options. These hedging strategies are a generalization of static hedging. In addition, the result is valid for all shapes of payoff, including path-dependent. Two cases of hedging methods are studied. The first one, as in Leland, assumes rehedging takes place at fixed time intervals. The second one supposes rehedging takes place when the delta moves by more than a fixed proportion. The pricing of securities in that frame can be done by solving a non-linear partial differential equation, and optimal hedging strategies, using various kinds of options, can be found so as to minimize transaction costs. The convergence result for the replication strategy is shown in detail.

The seventh chapter introduces a new sort of financial derivatives, which we christen "switch options". These products can hedge the risk related to the acquisition or to the business risk of an investment project. We define Switch options as path-dependent derivatives written on a single underlying that are activated every time the underlying hits a barrier and deactivated every time it hits another barrier. At maturity, if the option is activated, the holder receives a payoff that is a function of the underlying at that time; if it is not activated, the payoff is a different and lower function of the underlying's price. The number of times such an option can be activated and deactivated is not bounded. Unlike a standard barrier option, the Switch option is never totally cancelled when the underlying hits the barrier, as there is always a chance it will go back and hit the other barrier. In this chapter we will first focus on the relationship between real options and Switch options. We also price these options and compare them with standard barrier options. The technical analysis makes use of the Brownian Meander, and in particular we derive the joint law of the Brownian Meander and its maximum.

The eighth chapter focuses on the behavior of an informed investor. In this chapter we address the issue of quantifying the incentive to invest or disinvest from an equity investment to benefit from discrepancies between its real value and its market price. There exists an "insider option" for informed agents: the option to arbitrage the market price based on privileged information about the firm's projects. The decision of when to invest or disinvest and how much is indeed a real option based on market conditions. The exercise of such an option entails an effect on market prices. In this chapter we study the optimal arbitrage transactions an informed agent carries out and their influence on the market price. We model the discrepancy between the market price and the real value, known to the informed agent (maybe with some noise), and the impact on the market price of the trading strategy that maximizes the agent's wealth. The effect of a transaction on the price of a security determines how much it costs to trade this security, as well as the evolution of this price, which conditions future trading gains. We focus on the particular case of a manager trading his company's own stock. An existing models for the impact of transactions on prices is extended to the case of discrete transactions. It is derived from simple assumptions on the behaviour of market participants. A probabilistic approach is proposed to determine the optimal control applied to the market price by the informed agent. Analytical solutions are derived to calculate the value of "realigning the price" for an informed market participant, and the properties of the controlled market price are discussed.

Finally, the ninth chapter presents some concluding remarks.

Chapter 2 DECISION DELAYS AND EMBEDDED OPTIONS[1]

Since the early 80's, advances in the real options literature have completely changed the way we evaluate investment opportunities. As shown in this literature, firms should not invest in projects which are expected to earn only the opportunity cost of capital. Managers can make choices about the project's characteristics and this flexibility creates embedded options. These options add value to the project and invalidate the traditional net present value (NPV) rule. Among them, we can quote the option to defer the investment spending (McDonald and Siegel (1986)), the option to abandon an active project (Majd and Myers (1990)), the option to expand or to reduce the production capacity (Abel and Eberly (1996)) or the option to choose the production technology (He and Pindyck (1992)).

Although this literature has made a great step toward a better understanding of investment decisions, little research has focused on the practical side of the investment spending. One of the major characteristics of the capital budgeting process is the delay existing between the investment decision and its implementation. This implementation delay is generally associated with the decision process within the firm or the gathering of the financing funds necessary to undertake the investment spending. Harris and Raviv (1996), citing Taggart (1987), assert that projects are generally initiated from the bottom up, suggesting a centralization of the capital allocation process. Depending on the nature and the size of the investment, projects that have been approved at the division level may have to be submitted to headquarters. Although all these intermediary steps take time, this point has been ignored without exception in the real options literature whereas it can have important consequences. Depending on the evolution of the decision variable during this implementation lag, the investment opportunity may have lost part of its attractiveness.

Beyond the analysis of the effects of capital budgeting practices within firms, typical applications of our model include the services offered by specialized investment funds or the cost of the recourse to outside financing. For large projects it often takes considerable time to gather all the financing funds as the decision process within the institutions involved can be highly time consuming. For example, the time necessary to gather investors for a closed-end investment fund typically reaches one year during which the economic and market conditions can completely change. The type of financing funds may have an impact over the investment policy of firms as they condition the availability of the option to abandon investment opportunities during the implementation delay. The use of outside funds, which typically destroys such options for reputation concerns in a classic manager-investor conflict as described in Jensen and Meckling (1976), reduces the value of the investment opportunity. There is nevertheless room for negotiation: if funds are raised externally the firm can pay the right to cancel the investment process, depending on the evolution of the decision variable. In the same way, the services offered by specialized investment funds constitute a typical application of these options. Investment opportunities on emerging markets can take time to be

[1] THE ORIGINAL VERSION OF THIS CHAPTER WAS CO-WRITTEN WITH ERWAN MORELLEC AND IS TO APPEAR IN *REAL OPTIONS AND INVESTMENT UNDER UNCERTAINTY*, EDS. E. SCHWARTZ AND L. TRIGEORGIS, MIT PRESS.

realized and an investment specialist can reduce the delay between the investment decision and its real implementation by providing a dedicated vehicle. The best example in that case is an open-ended fund. The price to pay for the immediacy of the opportunity is reflected in the bid-ask spread for the fund as opposed to a closed-end investment which prevents the early withdrawal of funds.

This paper analyses in a single unifying framework the valuation and behavioral consequences of the real options' implementation delay. We use the computation of first passage times[2] to derive closed-form formulas relating the value of the investment opportunity and the investment threshold to the size of the delay. We find that the implementation delay can reduce the value of the investment opportunity by a large amount when there are no "minimum profitability requirements" once the investment decision has been taken. This result shows that the delay creates by itself the embedded option to abandon the investment project during its implementation should the decision variable evolve unfavorably. If the option is readily available, its value equals the increase in expected payoffs it permits. When the availability of this option depends on outside parties, its value constitutes the maximum price at which the agent holding the investment opportunity will be willing to negotiate it.

Using this general valuation framework, we compute the value of this option under alternative exercise policies corresponding to different minimum profitability requirements during the implementation period. We show that profitability requirements generate a higher project value only if they apply to the whole implementation lag. Minimum profitability requirements at the implementation date do not increase the welfare of investors when the investment decision is taken optimally. Therefore, it is not optimal for the headquarters of a firm to force the operational division to invest only if the decision variable is above a new cutoff level at the end of the implementation delay. In the same way, there is no interest for the investing firm to negotiate with its partners a covenant allowing for its withdrawal from the project when all the financing funds are gathered. On the contrary we find that "American" abandonment options increase the value of the investment project. Considering the difficulty of writing and enforcing contracts, we derive a so called "Parisian[3]" abandonment rule that can be implemented at a low writing cost on the contrary to the value maximizing abandonment rule.

In the following section, we review the traditional approach where there is no implementation delay and the value of an investment opportunity is the solution to a free boundary problem similar to that of an American financial option. In section two, we use a general valuation framework to describe the effects of the implementation delay on investment decisions. Section three provides the optimal value of the investment opportunity under various abandonment policies associated with this implementation delay. Section four concludes the paper and presents possible extensions.

[2]First passage times have already been used in the paper by Mauer and Ott (1995) in order to compute the mean replacement time of corporate assets. To our knowledge, our paper is the first that makes a systematic use of them. This singularity is due to the difficulty associated with the resolution of highly non linear ODE.

[3]See Chapter 3 and/or Chesney, Jeanblanc and Yor (1997) for a detailed presentation of Parisian options.

The traditional approach

We study the investment decision of a firm in a stochastic environment. At any time t the firm can invest in a project yielding an operating profit that depends on a decision variable $(S_t, t \geq 0)$ ruled by the diffusion process

$$\begin{cases} dS_t = S_t\left(\mu dt + \sigma dZ_t\right) \\ S_0 = x \end{cases} \tag{2.1}$$

where μ and σ are constant and $(Z_t, t \geq 0)$ is a Brownian motion defined on a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$[4]. $\mathcal{F}_t$ represents the information available at time t.

The literature on real options describes the investment decisions of firm thanks to two technics closely related to each other: dynamic programming and contingent claims analysis. These technics essentially differ about the assumptions they involve concerning investors, the financial markets and the discount rates used by investors. Due to these differences, the results they yield are similar although not identical[5].

One common characteristic of these models is the implicit assumption that actions are taken instantaneously: the project is started as soon as the investor has decided to invest. Using the stationary property and the Markovian property of the cash flows generated by the project, traditional real options models consider that the investment starts as soon as the decision variable $(S_t, t \geq 0)$ hits some constant optimal level[6].

The first valuation method used in the real options literature amounts to finding the optimization program of an investor through dynamic programming arguments. This investor is generally risk neutral and has rational expectations. He maximizes the present value of the cash flow generated by the investment through an appropriate timing of the investment decision. Let us denote $V(S_t, t)$ the value of the investment project, $f(S_t, t)$ the bounded monotonic profit flow function, c^+ (respectively c^-) the investment cost for a positive (respectively negative) investment, and ρ the investor's required rate of return i.e. the opportunity cost of capital[7]. The maximization program of this investor over an infinite time horizon can be written

$$V(x,t) = \max_{\{h^*,l^*\}} \mathbb{E}\left\{\int_0^{+\infty} e^{-\rho s}\left(f\left(S_s^{x,t}, s\right)ds - c^+ dP_{t+s} + c^- dN_{t+s}\right)\right\}$$

where P_t (N_t) is a non-decreasing (non-decreasing) function of time representing the cumulation of positive (negative) investment spending up to time t, and (h^*, l^*) is the pair of values of the underlying variable corresponding to optimal investment decisions. The Bellman equation associated with this optimization problem in the continuation region is[8]

$$\frac{1}{dt}\mathbb{E}\left[dV(S_t, t)\right] + f(S_t, t) = \rho V(S_t, t) \text{ for } S_t \in \left]l^*, h^*\right[\tag{2.2}$$

[4] $(\mathcal{F}_t, t \geq 0)$ is the filtration generated by the Brownian motion

[5] See Harchaoui and Lasserre (1995).

[6] Dixit and Pyndick (1994) provide a detailed presentation of this principle.

[7] If the investor discounts all its future cash flows at a constant rate ρ then we must have $\rho > \mu$ for the expected present value of the payoffs generated by the project, $\mathbb{E}\left\{\int_t^\infty \exp\left(-\rho(u-t)\right) S_u du / \mathcal{F}_t\right\}$, to be finite at any time t, $t \geq 0$ when f is the identity.

[8] See Duffie (1988) chapter 23 or Dixit and Pindyck (1994) chapter 4.

subject to familiar value-matching and smooth-pasting conditions[9] at $S = h^*$ and $S = l^*$. The LHS of this equation is the expected return on the investment project whereas the RHS is the investor's required return. The solution of this equation yields the optimal investment thresholds h^* and l^* and the value of the investment opportunity.

The other valuation method relies on an analogy between real and financial investment decisions. The firm has an option to invest in a project and the value of this option can be found thanks to the usual contingent claim valuation framework. By an application of Theorem 3 of Cox, Ingersoll and Ross (1985), the value of the investment opportunity satisfies the following fundamental valuation equation

$$rV(S_t,t) = \frac{1}{2dt}V_{SS}(S_t,t)\,d\langle S,S\rangle_t + \frac{1}{dt}V_S(S_t,t)\left(\mathbb{E}[dS_t] - \lambda(S_t,t)\,dt\right) + f(S_t,t)$$

where r is the continuously compound risk-free interest rate and $\mathbb{E}[dS_t] - \lambda(S_t,t)$ is the risk adjusted [10] drift of the underlying variable $(S_t, t \geq 0)$. In order to find $\lambda(S_t,t)$ capital markets must be complete: there must exist an asset or a dynamic portfolio of assets spanning the stochastic changes in the value function $V(S_t,t)$. The solution of this equation[11] gives the value of the investment opportunity and the optimal exercise boundary of this option [12]. Note that if the investment is perpetual and if f is independent of time, then the value function does not depend on time. In this case, the partial differential equation above becomes an ordinary differential equation.

General model of delayed investment decisions

In this subsection, we build a general valuation framework for the value of the investment opportunity when there is a delay between the investment decision and its implementation. Our analysis significantly differs from the "time to build"[13] literature as our implementation delay accounts for the lag existing between the decision to invest and the spending of the first dollar by the firm, not for the speed at which production facilities can be built.

As mentioned earlier, the implementation delay can be due to the research of an investment opportunity on an emerging market, to the capital budgeting process within the firm or to the time spent gathering the financing funds. Although our model applies to a wide range of implementation delays, we will emphasize in the following section the role of the capital budgeting process in altering the value of the investment opportunity. In order to keep the presentation simple, we will take the simplest environment possible. As in Harris and Raviv (1996), our firm is composed of headquarters and a single division[14]. The investment decision is initiated by the division manager but he must obtain capital from the

[9] When the optimization program is strictly concave, i.e. when the indirect expected discounted payoff $V(.,.)$ is strictly concave, the boundary conditions ensuring that we are along the optimal path are called high-contact conditions and involve second order derivatives. See Dumas (1991) for a good exposition of value-matching, smooth-pasting and high-contact conditions.

[10] $\lambda(S_t,.)$ is the risk premium associated with $(S_t, t \geq 0)$.

[11] The boundary conditions used to solve this equation are the same than those used to solve equation (2).

[12] When investment is reversible there are two exercise boundaries, one for investment and one for disinvestment.

[13] In this literature each dollar invested in the investment opportunity gives the firm the option to spend another dollar in the project. See for example Majd and Pindyck (1987).

[14] Considering that the implementation delay reduces the value of the investment opportunity, the division would probably do better of as a stand alone entity. We do not address this issue

headquarters. This decentralization of the investment decision is due to the specific human capital of the division manager. We assume that he has no incentive to misrepresent his information but that the information transfer within the firm and the decisions concerning the capital allocation take time. We will focus on discretionary investments for which there is generally such a bottom up process.

In our setting agents are risk neutral and the firm has an investment opportunity in a non-traded asset yielding stochastic returns. Markets are incomplete in the sense that it is impossible to buy an asset or a dynamic portfolio of assets spanning the stochastic changes in the value of the project. There is no futures market either for the decision variable or the size of the investment project prevents the firm from taking a position on such a market. Moreover, we consider that the project, once installed, goes on producing output forever[15].

We will be interested in the value of the investment opportunity when the profit function associated with an active project has the following special form[16] $f(x) = \psi x^{\gamma}$ where $\gamma > 0$ and ψ does not depend on x. We denote $F(S_t, \infty)$ the expected present value of future profits when the investment spending is realized at time t i.e.

$$F(S_t, \infty) = \int_t^{\infty} ds e^{-\rho(s-t)} \mathbb{E}_{S_t}[f(S_s)] = \Delta S_t^{\gamma}$$

with

$$\Delta = \frac{\psi}{\left[\rho - \gamma\left(\mu + \frac{1}{2}\sigma^2(\gamma - 1)\right)\right]}$$

The Markovian features of the standard model and the stationary property of the distribution of the payoffs generated by the active project imply that the value of the investment project depends on time only through the time dependence of the decision variable $(S_t, t \geq 0)$. Consequently, the investment decision will occur at the first instant when this variable hits some *constant* optimal investment threshold h^*.

Let us define for an arbitrary investment boundary h the stopping time $T_h(S)$ by

$$T_h(S) = \inf\{s \geq 0,\ S_s = h\}$$

When we take a delay-related constraint into account, the investment is realized at a parameterized stopping time $\theta(S_t - S_{T_h}, t \geq T_h)$ independent of $T_h(S)$ and of $(S_t, t \leq T_h(S))$ such that the level of the decision variable at time θ is independent of θ and $T_h(S)$; in other words S_θ is independent from θ and T_h. This time can be viewed as a general constraint. It can be a fixed time or any time that conditions

as in most economic organizations the capital allocation is centralized as in our analysis (see for example Ross (1986)). The paper by Harris and Raviv provides a rationale for this capital budgeting process relying on information and incentive problems within firms.

[15]Standard justifications of the assumption of irreversibility rely on the lemons problem or capital specificity of the assets in place (see for example Abel and alii (1995)). The irreversibility assumption is very realistic for economic activities which are highly capital intensive such as mining projects or offshore petroleum leases. Indeed for such activities, it is unusual to observe temporary shut down or capacity reduction. Dias (1997) remarks "This kind of investment has a high degree of irreversibility. For example, the drilling of a well is completely irreversible [...]".

[16]The analysis could easily be extended to other specifications for the profit function. Nevertheless, this specification is general enough to allow us to treat most of the traditional financial or economic applications. For example, $f(R_t, \overline{K}_t) = \psi R_t^{\gamma} \overline{K}_t^{1-\gamma}$ where R_t represents the level of a demand shock and $\overline{K}_t$ is the production capacity at time t can account for a firm with a CRS Cobb-Douglas production function facing an isoelastic demand curve (this kind of specification can be found in Abel and Eberly (1996) or Dixit (1991)).

the implementation of the investment spending. We require that θ be independent from S_θ to simplify the calculations.

If we denote C_e the direct investment cost, the value of the investment project for $S_0 < h$ is given by[17]

$$V\left(S_0, h, \Delta, C_e, \theta\right) = \mathbb{E}_{S_0}\left[\int_0^\infty dt e^{-\rho t} f\left(S_t\right) \mathbb{I}_{t \geq \theta}\right] - C_e \mathbb{E}_{S_0}\left[e^{-\rho\theta}\right]$$

where the first term of the RHS is the present value of expected profits generated by the investment project and C_e isd the direct investment cost. Using the independence between $(S_{t+\theta} - S_\theta, t \geq 0)$ and $(S_t, t \leq \theta)$, we can write the value of the investment opportunity as

$$V\left(S_0, h, \Delta, C_e, \theta\right) = \mathbb{E}_{S_0}\left[e^{-\rho\theta}\left(F\left(S_\theta, \infty\right) - C_e\right)\right]$$

by the Strong Markov Property. Now, let us write $\theta = (\theta - T_h) + T_h$. Thanks to the independence before and after T_h and standard results concerning first passage times[18] , we get

$$\begin{aligned}
V\left(S_0, h, \Delta, C_e, \theta\right) &= \mathbb{E}_{S_0}\left[e^{-\rho\theta}\Delta S_\theta^\gamma\right] - C_e \mathbb{E}_{S_0}\left[e^{-\rho\theta}\right] \\
&= \mathbb{E}_{S_0}\left[\Delta S_\theta^\gamma\right] \mathbb{E}_{S_0}\left[e^{-\rho T_h}\right] \mathbb{E}_h\left[e^{-\rho(\theta - T_h)}\right] - C_e \mathbb{E}_{S_0}\left[e^{-\rho T_h}\right] \mathbb{E}_h\left[e^{-\rho(\theta - T_h)}\right] \\
&= \left(\frac{S_0}{h}\right)^{\xi_1} A\left(\theta\right)\left(\Delta h^\gamma B\left(\theta\right) - C_e\right) \qquad (2.3)
\end{aligned}$$

with $\xi_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\frac{2\rho}{\sigma^2} + \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2}$, $A\left(\theta\right) = \mathbb{E}_h\left[e^{-\rho(\theta - T_h)}\right]$ and $B\left(\theta\right) = \mathbb{E}_h\left[\left(\frac{S_\theta}{h}\right)^\gamma\right]$. $A\left(\theta\right)$ is the discounting factor associated with the implementation lag while $B\left(\theta\right)$ accounts for the exponential of the change in the expected operating profit due to the path of the decision variable during this delay.

Using equation 2.3, we can write

$$V\left(S_0, h, \Delta, C_e, \theta\right) = V\left(S_0, h, \Delta A\left(\theta\right) B\left(\theta\right), C_e A\left(\theta\right), 0\right) \qquad (2.4)$$

The value of the investment opportunity, for a given investment barrier h, can be expressed as the value of the investment opportunity with no delay and modified parameters. Straightforward calculations give us the optimal barrier and the optimal value of the investment opportunity.

Proposition 1 *When there is an implementation delay and the instantaneous profit function associated with an active project is $f\left(x\right) = \psi x^\gamma$, then the value of the investment opportunity and the optimal investment threshold are respectively given by*

$$V\left(S_0, \Delta, C_e, \theta\right) = A\left(\theta\right) B\left(\theta\right)^{\frac{\xi_1}{\gamma}} \gamma S_0^{\xi_1}\left(\frac{C_e}{\xi_1 - \gamma}\right)^{\frac{\gamma - \xi_1}{\gamma}}\left(\frac{\Delta}{\xi_1}\right)^{\frac{\xi_1}{\gamma}} \qquad (2.5)$$

and

$$h^*\left(\Delta, C_e, \theta\right) = \left(\frac{\xi_1}{\left(\xi_1 - \gamma\right)\Delta} C_e\right)^{\frac{1}{\gamma}}\left(B\left(\theta\right)\right)^{-\frac{1}{\gamma}}.$$

[17] Hereafter, we write T_h for $T_h(S)$ and θ for $\theta(S)$.

[18] For an heuristic presentation of these results see Dixit and Pindyck (1994) pp.316. A rigorous proof can be found in Karatzas and Shreve (1991) pp.196.

From 2.5 we can easily obtain the ratio of the value of the investment opportunities at their respective optima with a delay to that with no delay as

$$r(\theta) = A(\theta)\, B(\theta)^{\frac{\xi_1}{\gamma}} = \mathbb{E}_{h^*}\left[e^{-\rho(\theta - T_h)}\right]\left(\mathbb{E}_{h^*}\left[\left(\frac{S_\theta}{h^*}\right)^{\gamma}\right]\right)^{\frac{\xi_1}{\gamma}} \tag{2.6}$$

This ratio allows an easy comparison among different constraints over the implementation period. It appears that the ratio combines two effects: the earlier the investment time is, the greater the ratio. But on the other hand, the lower the level of S at that time, and the smaller the ratio.

In the following section we show that the implementation delay has a significant impact on the value of investment opportunities. We provide alternative investment rules so as to minimize the associated value reduction.

Implementation delay and embedded options

We have seen in the previous section that any investment opportunity with an implementation delay can be valued according to equation 2.5 from proposition 1. We now turn to specific applications concerning the abandonment option associated with the implementation lag. Although we do not focus on this aspect of investment decisions, the availability of the abandonment option depends on the type of financing funds used by the holder of the investment project. Indeed, if the project is financed internally, then this option always exists and we will see that, depending on the exercise policy followed, it can be interpreted in terms of profitability requirements. On the contrary, the use of external funds links the manager to outside investors and for reputation concerns associated with the classic manager-investors conflict described by Jensen and Meckling (1976), he may be forced to invest in this investment project. This phenomenon underlines a new type of financing costs whose magnitude depends on the type of funds used and the liquidity of the financing markets.

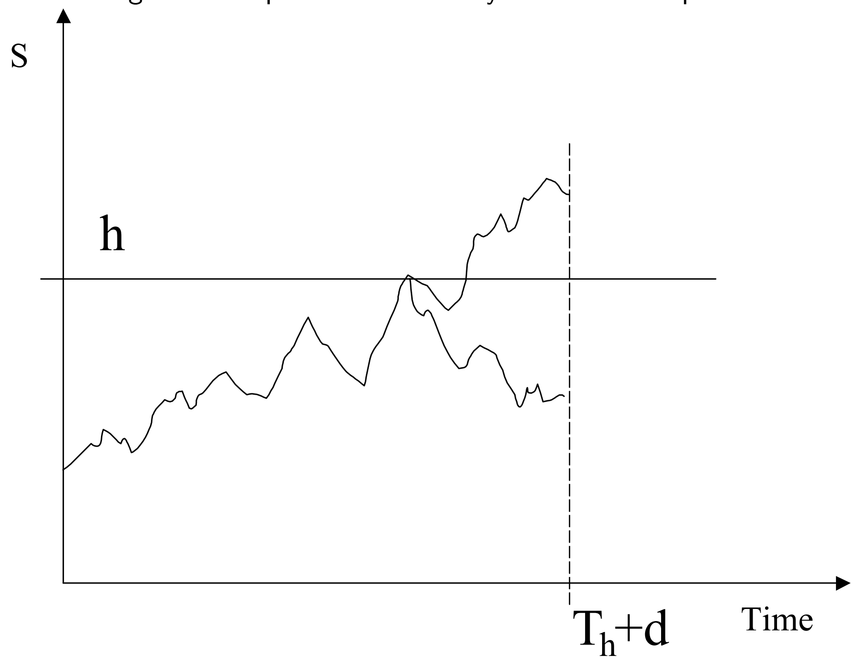
With respect to the abandonment of the project, various type of options can be considered[19]. All of these options can be interpreted in terms of profitability requirement by the headquarters or the operational division. The most obvious one is a European abandonment option giving the investor the right to leave the project at time θ should the decision variable evolve unfavorably. Alternatives are to give him the option to cancel the whole thing during the interval $[T_h, \theta]$ if the decision variable falls beneath some prespecified level. We will see American options allow the investing firm to get a higher expected payoff from the project whereas the European abandonment option is valueless.

Implementation delay with no exit option:

Let us consider first the case where the delay is a fixed time d and the operational manager invests at time $T_h(S) + d$ whatever the evolution of the decision variable during the time interval $[T_h(S), T_h(S) + d]$ (in this case $\theta = d + T_h$). This simple case gives us the value reduction associated with the implementation lag when there is no abandonment option. In this application, neither the operational division nor the headquarters have any profitability requirement once the investment decision as been taken optimally at $T_h(S)$. The delay with no exit option is illustrated in Figure 1 on p. 24.

[19] The extensive proofs of the results given in this section are given in Chapter 3.

Figure 1 Implementation Delay and no Exit Option



Straightforward calculations give

$$\begin{aligned} A(\theta) &= e^{-\rho d} \\ B(\theta) &= e^{\gamma d\left(\mu - \frac{\sigma^2}{2}(1-\gamma)\right)} \end{aligned}$$

and the optimal investment boundary is

$$h^*(\Delta, C_e, d) = e^{-d\left(\mu - \frac{\sigma^2}{2}(1-\gamma)\right)} h^*(\Delta, C_e, 0)$$

This investment boundary is strictly lower than $h^*(\Delta, C_e, 0)$ when $\gamma > 0$ if we assume $\mu \geq \frac{\sigma^2}{2}$. When there is an implementation delay, investors anticipate investment decisions by choosing an investment threshold lower than in the standard analysis. If we had $\mu \leq \frac{\sigma^2}{2}$, then we would expect to find a higher threshold.

Using equation 2.6, we see that the ratio of the value of the investment opportunity with delay to that with no delay is

$$r(\theta) = \exp\left\{ d\xi_1 \left(\mu + \frac{\sigma^2(\gamma - 1)}{2} - \frac{\rho}{\xi_1} \right) \right\}.$$

Numerical simulations reveal that the implementation delay has an impact on the value of the investment opportunity. with reasonable parameters[20] , we find that the ratio is about 94%. By changing the drift μ and increasing it to 5%, the ratio jumps to 98%. This illustrates how the delay can hurt the value of the investment if the state variable does not tend to increase significantly over the waiting period on average. The associated value reduction is due to the discounting of the investment NPV from T_h to $T_h + d$ and to the dependence of the value of the project on the evolution of the decision variable during the implementation

[20] $d = 1$; $\rho = 0.075$; $\mu = 0.01$; $\sigma = 0.20$; $\gamma = 0.7$

period. The agent in charge with the investment decision does not benefit from the opportunity to abandon the project during the implementation lag. Furthermore, since markets are incomplete, it is impossible for the firm to suppress the uncertainty concerning the evolution of the value of the decision variable during this period of time. Therefore, the value of the investment opportunity depends on the evolution of the decision variable[21] over the time interval $[T_h, T_h + \theta]$.

One can show that the value reduction due to the implementation delay is increasing as a function of the opportunity cost of capital for reasonable parameter values. As capital gets more costly, the effect of the discounting factor associated with the implementation period and the uncertainty concerning the profitability of the project weight more on the value of the investment opportunity.

European abandonment option

This application accounts for the case of an investment decision taken at division level that has to be approved by the headquarters. If the approval arrives d units of time after the submittance, then the operational manager invests only if the state variable is above a new cutoff level h'. The headquarters have no own profitability requirements during the implementation lag and the approval is based on the strategical merits of the project (new line of business, expanded markets). On the contrary, there are profitability requirements when the operational manager is in charge with the decision i.e. at times T_h and $T_h + d$.

The variable θ, that is the time when the investment is decided, is here the first instant when d units of time after having hit h, the state variable is above h', such that if it is not over h', one has to wait until the process hits h again before re-waiting until it hits h again, and so on. Optimally, h' is obviously expected to be lower than h. Using the definitions from Chapter 3, we have $\theta = \nu_d^{h',h}(S)$. In Chapter 3, we show that $S_{\nu_d^{h',h}}$ is independent of $\nu_d^{h',h}(S)$. The European implementation delay is shown in Figure 2 on p. 26.

Chapter 3 formally defines this stopping time and provides the tools to compute the law of this stopping time. Using Proposition 4 on p. 39, we get

$$\begin{aligned} A(\theta) &= \frac{e^{-\rho d}\left(1 - \mathcal{N}\left(b\sqrt{d} + \ln\left(\frac{h'}{h}\right)\left(\sigma\sqrt{d}\right)^{-1}\right)\right)}{1 - \mathcal{N}\left(-\sqrt{(2\rho + b^2)\, d} + \ln\left(\frac{h'}{h}\right)\left(\sigma\sqrt{d}\right)^{-1}\right)} \\ B(\theta) &= e^{d\gamma\left(\mu - \frac{\sigma^2}{2}(1-\gamma)\right)} \frac{1 - \mathcal{N}\left((b + \gamma\sigma)\sqrt{d} + \ln\left(\frac{h'}{h}\right)\left(\sigma\sqrt{d}\right)^{-1}\right)}{1 - \mathcal{N}\left(-b\sqrt{d} + \ln\left(\frac{h'}{h}\right)\left(\sigma\sqrt{d}\right)^{-1}\right)}. \end{aligned}$$

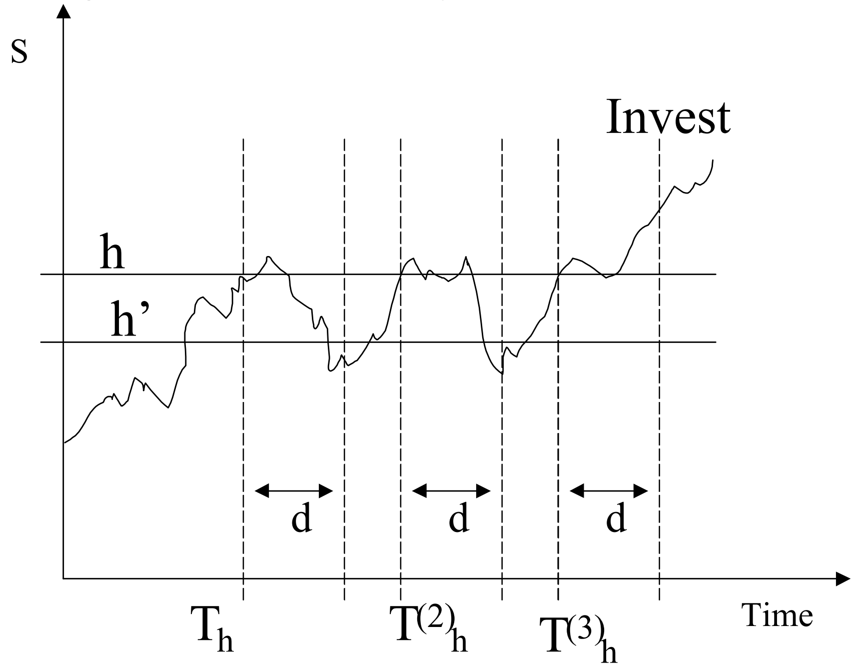
where $\mathcal{N}$ is the Standard Normal cumulative distribution function

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt, x \in \mathbb{R}$$

and $b = \frac{\mu - \frac{\sigma^2}{2}}{\sigma}$. We can find the ratio of the optimal value of the investment project with European abandonment option to the value of the investment project with

[21] The mean change in the decision variable during the implementation delay is given by $e^{\gamma d\left(\mu - \frac{\sigma^2}{2}(1-\gamma)\right)}$ which is larger than one. Nevertheless, as the real path followed by the stochastic process ruling the evolution of this variable can be unfavorable to the firm, the abandonment options are not valueless.

Figure 2 Implementation Delay with European Exit Option



no delay:

$$r(\theta) = \frac{e^{-\rho d}\left(1-\mathcal{N}\left(b\sqrt{d}+\ln\left(\frac{h'}{h}\right)\left(\sigma\sqrt{d}\right)^{-1}\right)\right)}{1-\mathcal{N}\left(-\sqrt{(2\rho+b^2)\,d}+\ln\left(\frac{h'}{h}\right)\left(\sigma\sqrt{d}\right)^{-1}\right)}$$
$$\left(e^{d\gamma\left(\mu-\frac{\sigma^2}{2}(1-\gamma)\right)}\frac{1-\mathcal{N}\left((b+\gamma\sigma)\sqrt{d}+\ln\left(\frac{h'}{h}\right)\left(\sigma\sqrt{d}\right)^{-1}\right)}{1-\mathcal{N}\left(-b\sqrt{d}+\ln\left(\frac{h'}{h}\right)\left(\sigma\sqrt{d}\right)^{-1}\right)}\right)^{\frac{\xi_1}{\gamma}}.$$

Numerical results show that this ratio is decreasing in h'/h. The European abandonment option will always be exercised as we have $h'/h = 0$ at the optimum. When the manager holds a European abandonment option, the value of waiting to invest at the maturity of this security (i.e. at the implementation date) is lower than the benefits of investing directly. Postponing the investment spending would lower the value of the profits generated by the investment project by an amount larger than an immediate exercise at $T_h + d$ would. Table 2.1 on p. 27 shows numerical results with the same parameters as in the preceding example.

As the European abandonment option is valueless, it is not optimal for the headquarters, once they have approved the project on its strategical merits, to have any profitability requirement at $T_h + d$ if the investment decision has been taken optimally at T_h. In the same way, their is no interest for the firm to negotiate with external investors the availability of this exit option at the end of the gathering of the funds when the investment project requires external financing.

The Parisian abandonment option

We give here the value of the investment opportunity when the manager invests at θ only if the decision variable reaches a prespecified level and remains above

$\frac{h'}{h}$	$r(\theta) = \frac{\text{European exit option}}{\text{Value with no delay}}$	$\frac{h'}{h}$	$r(\theta) = \frac{\text{European exit option}}{\text{Value with no delay}}$
0.0	.93595	0.7	.90732
0.1	.93595	0.8	.85047
0.2	.93595	0.9	.76597
0.3	.93595	1.0	.67133
0.4	.93595	1.1	.58042
0.5	.93595	1.2	.49971
0.6	.9310	1.3	.43073

Table 2.1 Valuation of the European Entry Option

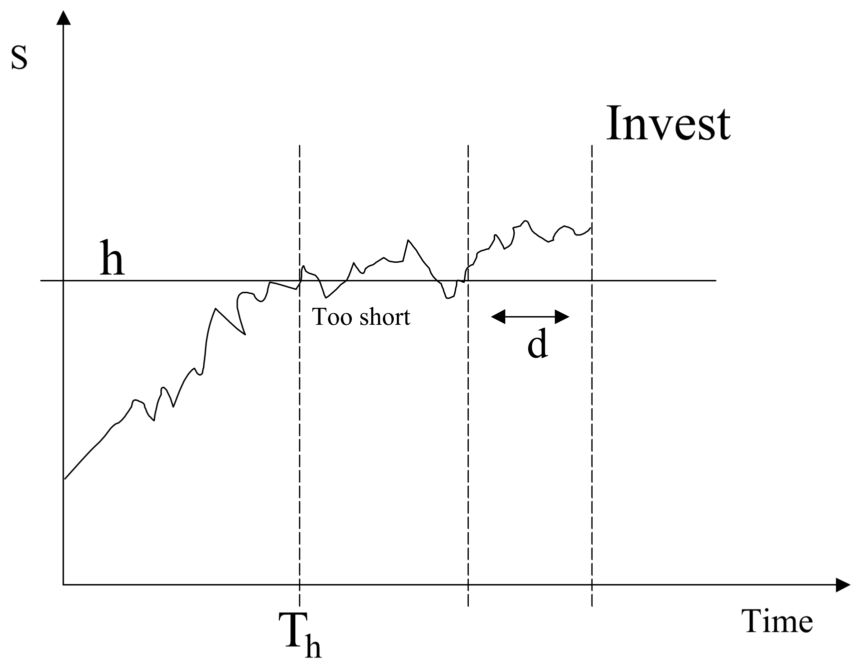


Figure 3 Implementation Delay with Parisian Option

this level for a time interval longer than a fixed amount of time (the window) without going back down. In this application, the time window is equal to the implementation delay and the prespecified level is set at an optimal value, the best investment threshold h^*. The decision triggering criterium is a so-called Parisian stopping time depending on the size of excursions of the state variable over the optimal investment threshold. We have $\theta = H^+_{h,d}(S)$, where $H^+_{h,d}(S)$ is defined below. The Parisian abandonment option is represented graphically in Figure 3 on p. 27.

This investment criterion reflects the will of the investing firm (the headquarters and the operational division) to check that the market conditions remain favorable during the implementation lag. The firm has profitability requirements which are constant over time i.e. the level of the state variable has to stay above some constant cutoff level during the whole implementation lag. Intuitively, this cutoff level will be lower than the investment boundary of the standard case without implementation lag[22].

[22] Note that this Parisian investment criterion is not absolutely optimal. It only uses one degree of freedom, that is the level of the barrier.

In order to find the value of the investment opportunity under the Parisian investment policy, we define the following random variables, all linked to a random process X:

$$\begin{aligned} g_t^a(X) &= \sup\{s \leq t : X_s = a\} \\ H_{a,d}^+(X) &= \inf\{t \geq 0 : (t - g_t^a(X)) \geq d, X_t \geq a\}. \end{aligned}$$

$g_t^a(X)$ represents the last time the process X crossed the level a. It can be checked it is not a stopping time for the Brownian filtration $\mathcal{F}_t$, but for the slow filtration $\mathcal{G} = (\sigma(sgn(B_t)) \vee \mathcal{F}_{g_t})_{t\geq 0}$ which represents the information on the Brownian motion until its last zero plus the knowledge of its sign after this[23]. $H_{h,d}^+(X)$ is therefore the first instant when the process has spent d units of time consecutively over the level a.

Using the above notations, the Parisian investment policy is described by the stopping time $H_{h,d}^+(S)$. In this case, the value of the investment opportunity can be written

$$V^P(S_0, h, d, C_e) = \mathbb{E}_{S_0}\left[e^{-\rho H_{h,d}^+(S)}\left(F\left(S_{H_{h,d}^+(S)}, \infty\right) - C_e\right)\right]$$

The Parisian criterion allows for two degrees of freedom: the barrier h and the "time window" d, which we naturally choose as the minimal implementation delay.

Following the approach we outlined for the general model, we write for $S_0 < h$, the value of the investment project as

$$V_P(S_0, h, \Delta, C_e, d) = \mathbb{E}_{S_0}\left[\int_0^\infty dt e^{-\rho t} f(S_t)\, \mathbb{I}_{t \geq H_{h,d}^+(S)}\right] - C_e \mathbb{E}_{S_0}\left[e^{-\rho H_{h,d}^+(S)}\right]$$

Using the independence of the index paths after and before $H_{h,d}^+(S)$, we can write the value of the investment opportunity as

$$V_P(S_0, h, \Delta, C_e, d) = \mathbb{E}_{S_0}\left[e^{-\rho H_{h,d}^+(S)}\left(F\left(S_{H_{h,d}^+(S)}, \infty\right) - C_e\right)\right]$$

Writing $S_t = S_0 \exp\left(\sigma Z_t^b\right)$ where $Z_t^b = B_t + bt$ and $b = \frac{\mu - \frac{\sigma^2}{2}}{\sigma}$, we have

$$V_P(S_0, h, \Delta, C_e, d) = \mathbb{E}_{S_0}\left[e^{-\rho H_{a,d}^+(Z^b)}\left(F\left(S_0 \exp\left(\sigma Z^b_{H_{a,d}^+(Z^b)}\right), \infty\right) - C_e\right)\right]$$

with $a = \frac{1}{\sigma}\ln\left(\frac{h}{S_0}\right)$. Thanks to the equality in law between $H_{a,d}^+(Z^b)$ and $H_{0,d}^+(Z^b) + T^a\left(\left(Z^b\right)'\right)$, for two independent copies Z^b and $\left(Z^b\right)'$, and thanks to the independence between $\left(Z_t^b, t \leq T^a\right)$ and $\left(Z_t^b, t \geq T^a\right)$ we get

$$V_P(S_0, h, \Delta, C_e, d) = \mathbb{E}_{S_0}\left[e^{-\rho H_{a,d}^+(Z^b)}\left(F\left(h \exp\left(\sigma Z^b_{H_{0,d}^+(Z^b)}\right), \infty\right) - C_e\right)\right]$$

Using the approach for the general model outlined in section 2 we have

$$\begin{aligned} V_P(S_0, h, \Delta, C_e, d) &= \left(\frac{S_0}{h}\right)^{\xi_1} \mathbb{E}_{S_0}\left[e^{-\rho H_{0,d}^+(Z^b)}\right] \\ &\quad \times \left(\mathbb{E}\left[F\left(h \exp\left(\sigma\left(bd + m_1\sqrt{d}\right)\right), \infty\right)\right] - C_e\right). \end{aligned}$$

[23] For details see Azéma and Yor (1989).

d	$r(\theta)$ Parisian	$r(\theta)$ No exit
0.0	1.0	1.0
0.25	.99931	.98359
0.50	.99333	.96745
0.75	.98464	.95157
1.0	.97415	.93595
1.5	.94968	.90548
2.0	.92237	.87601
3.0	.86401	.8199
10.0	.50804	.51587

Table 2.2 Valuation of the Parisian Entry Option

where m_1 is the Brownian meander taken at time 1. The difficulty in the above expression is to calculate the Laplace transform of the Parisian time $H^+_{0,d}(Z^b)$. Thanks to the results of Chesney, Jeanblanc and Yor (1997) on Parisian options, we are able to directly write this value. The law of $B_{H^+_{0,d}}$ was also derived in Chesney, Jeanblanc and Yor (1997) and is given in Chapter 3. Using theorem 7 on p.43 (taken from Chesney, Jeanblanc and Yor (1997)) and after straightforward simplifications, we finally have

$$\begin{aligned} V_P(S_0, h, \Delta, C_e, d) &= \left(\frac{S_0}{h}\right)^{\xi_1} \frac{\Phi\left(b\sqrt{d}\right)}{\Phi\left(\sqrt{d(2r+b^2)}\right)} \\ &\times \left[\Delta h^{\gamma} \frac{\Phi\left((\sigma\gamma + b)\sqrt{d}\right)}{\Phi\left(b\sqrt{d}\right)} - C_e\right]. \end{aligned}$$

where
$$\Phi(x) = \int_0^{+\infty} z \exp\left(zx - \frac{z^2}{2}\right) dz = 1 + \sqrt{2\pi} x e^{-\frac{x^2}{2}} \mathcal{N}(x).$$

This can also be written as

$$\begin{aligned} A(\theta) &= \frac{\Phi\left(b\sqrt{d}\right)}{\Phi\left(\sqrt{d(2\rho+b^2)}\right)} \\ B(\theta) &= \frac{\Phi\left((\sigma\gamma + b)\sqrt{d}\right)}{\Phi\left(b\sqrt{d}\right)}. \end{aligned}$$

The ratio of the value of the Parisian investment opportunity with respect to the investment opportunity with no delay, at their respective optima, is

$$r(\theta) = \frac{\Phi\left(b\sqrt{d}\right)}{\Phi\left(\sqrt{d(2\rho+b^2)}\right)} \left(\frac{\Phi\left((\sigma\gamma + b)\sqrt{d}\right)}{\left(\Phi\left(b\sqrt{d}\right)\right)}\right)^{\frac{\xi_1}{\gamma}}.$$

We can compare the value of the investment project with no delay, with a delay and no exit option (or a European option), and with a delay and a Parisian investmen criterion, see Table 2.2 on p. 29. We use the same parameters as in the preceding examples.

The value of the investment opportunity is shown to be higher when the investor has the opportunity to choose the Parisian investment criterion than in the standard case, for reasonable parameter values. This investment policy gives the investor the option to give up its investment opportunity if the decision variable goes below a prespecified level h^P during the implementation of the investment spending. This option is freely obtained when the firm uses internal funds to finance its investment opportunity. Therefore, the cost of outside funds is higher than traditionally assumed as they often prevent the management from exercising such embedded options. The value of the Parisian option gives us the maximum price that the firm will be willing to pay for this option to be available during the gathering of outside financing funds.

Numerical imulations indicate that when the delay becomes very long, it becomes better to follow the European criterion, and invest as soon as the delay is expired. This leaves indeed a positive value for the project, while the Parisian criterion forces the investor to delay the investment too much: it becomes less and less likely that the state variable will spend consecutively a long period of time above the threshold.

When the investing firm holds a Parisian abandonment option, the optimal investment boundary is given by

$$h^P(\Delta, C_e, d) = h^*(\Delta, C_e, 0)\left(\frac{\Phi\left(b\sqrt{d}\right)}{\Phi\left((\sigma\gamma + b)\sqrt{d}\right)}\right)^{\frac{1}{\gamma}}$$

Since $\Phi(.)$ is a strictly increasing function, the investment boundary is lower than in the standard case with no implementation delay for $\gamma > 0$. One can notice that when $d = 0$, we find back the results associated with the standard case. Moreover, the larger the implementation lag, the lower the optimal investment barrier is and the lower the hurdle rate used by the firm. Note that if $\gamma = 1$, then the optimal barrier h^P in the Parisian case is lower with respect to the non-delayed case by a factor that corresponds to how higher the state variable should be at the time of investment. On average, the investment decision will intervene at the same level as in the non-delayed case.

This investment criterion provides the value of the decision variable under which their will be no investment. According to this model there exists a value of waiting to invest but the real investment threshold and the option premium can vary according to the basic parameters and the shape of the excursion of the decision variable over the barrier. The value of the decision variable for which investment will occur and the value of the option premium can therefore be over or under the standard ones. This phenomenon has recently been stressed by the empirical evidence concerning real options (see Quigg (1993)).

American abandonment option with an exponential exercise barrier

When we look at standard results concerning American options, it is clear that the optimal exercise barrier exhibits some time dependence. Therefore, the Parisian criterion is not absolutely optimal as for the corresponding option the abandonment barrier is constant through time. Indeed, if for example the state variable goes back under the barrier just one day before the end of the implementation, it would not be optimal to cancel it. As a matter of fact, there exists a time dependence of the optimal abandonment level to how much time the firm has spent in the implementation of its investment decision.

In this section, we give the value of the investment opportunity if the manager invests at θ only if the decision variable remains above an early abandonment

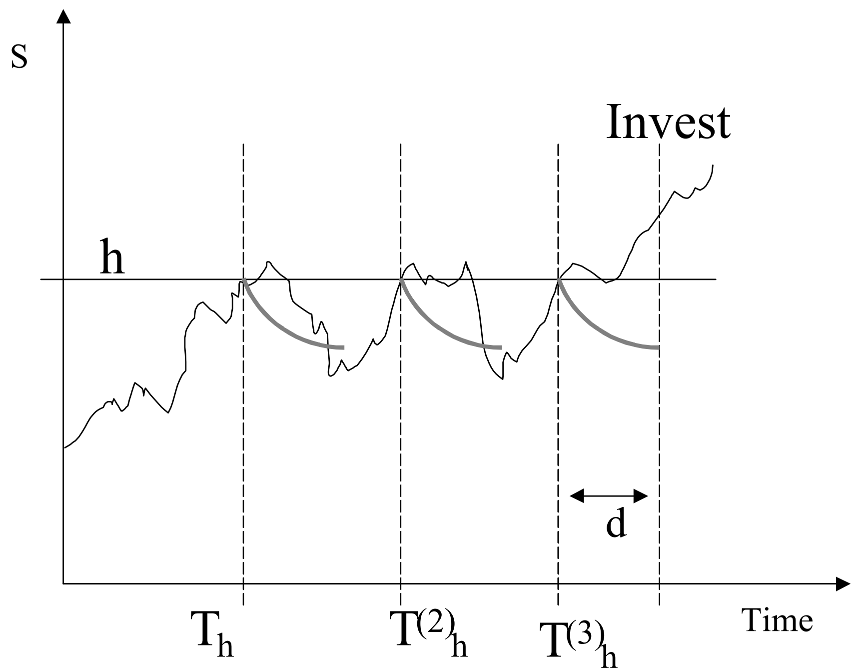


Figure 4 Implementation Delay with Exponential Exercise of the Exit Option

boundary depending on time during the implementation delay. In this case the headquarters take into account the cost of postponing once again the investment spending and have a decreasing minimum profitability requirement during the whole implementation lag. This case is a generalization of the Parisian criterion; the approach here is indeed "Parisian in concept", though the mathematical technique used is quite different. We have $\theta = \eta_d^{g,h}$, where $\eta_d^{g,h}$ is formally defined in Chapter 3. The exponential exit option exercise is graphed in Figure 4 on p. 31.

Following Omberg (1987), we approximate the early exercise boundary of the American abandonment option by an exponential function which is given the following form $g(t) = he^{-\sigma(\varepsilon+\beta t)}$ where ϵ and β are positive real numbers. $\epsilon > 0$ implies that there is an initial jump in the abandonment barrier while β takes into account the time dependence of this boundary.

Proposition 5 on p. 40 gives the value of the coefficients $A(\theta)$ and $B(\theta)$ when the manager follows this exercise policy, with $y(t) = -\varepsilon - \beta t$. We have

$$\begin{aligned} A(\theta) &= \mathbb{E}_0\left[e^{-\left(\rho+\frac{b^2}{2}\right)\eta_d^{y,0}(Z)}\right]\mathbb{E}_0\left[e^{bZ_{\eta_d^{y,0}}}\right] \\ B(\theta) &= \mathbb{E}_0\left[e^{-\frac{b^2}{2}\eta_d^{y,0}(Z)}\right]\mathbb{E}_0\left[e^{(b+\gamma\sigma)Z_{\eta_d^{y,0}(Z)}}\right] \end{aligned}$$

and for all positive λ and positive α

$$\mathbb{E}\left[e^{-\lambda\eta_d^{y,0}}\right] = \frac{e^{-\lambda d}\int_d^\infty dt\frac{\varepsilon\exp\left(-\frac{(\varepsilon+\beta t)^2}{2t}\right)}{\sqrt{2\pi t^3}}}{1-e^{-\varepsilon\sqrt{2\lambda}}\int_0^d dt\frac{\varepsilon\exp\left(-\left(\lambda+\beta\sqrt{2\lambda}\right)t-\frac{(\varepsilon+\beta t)^2}{2t}\right)}{\sqrt{2\pi t^3}}}$$

$$\mathbb{E}\left[e^{\alpha Z_{\eta_d^{y,0}}}\right] = \frac{\int_{-\infty}^{\infty} dz \frac{e^{\alpha z}}{\sqrt{2\pi d}} \left(e^{-\frac{z^2}{2d}} - e^{\beta z + \frac{\beta^2 d}{2} - \frac{(\varepsilon + |z+\varepsilon+\beta d|)^2}{2d}} \right)}{\int_d^{\infty} dt \frac{\varepsilon \exp\left(-\frac{(\varepsilon+\beta t)^2}{2t}\right)}{\sqrt{2\pi t^3}}}.$$

The optimal investment boundary can be obtained by the maximization of the value of the investment opportunity with respect to h, ϵ and β. We show in Chapter 3 how $\mathbb{E}\left[e^{-\lambda \eta_d^{y,0}}\right]$ converges towards the Parisian time, that is how

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[e^{-\lambda \eta_d^{y,0}}\right] = \mathbb{E}\left[e^{-\lambda H_{0,d}^{+}}\right]$$

with $\beta = 0$, which justifies the claim that this abandonment time generalizes the Parisian time.

Numerical simulations indicate that the abandonment option with exponential exercise barrier has a larger value than the Parisian abandonment option. This is what was expected as we have now three degrees of freedom h, ϵ and β instead of one. This stresses that there exists indeed a time dependence of the optimal abandonment level to the time already spent in the approval stage or to how long the firm has been consecutively gathering investors.

Concluding remarks

This chapter provides a general valuation framework for investment opportunities relying on the computation of first passage times. We show that the delay existing between the investment decision and its implementation has important valuation consequences. In particular, when the investing firm has no profitability requirement once the investment decision has been taken, the value of the investment opportunity can be reduced by almost 10%.

Although we focused on the capital budgeting process within the firm, the analysis can be readily extended to other applications. In particular the various types of options considered can constitute the basis for covenants in the contracts linking the investing firm to outside parties. The firm can negotiate the right with external investors to cancel the whole investment process if the evolution of the state variable is unfavorable. In the same way, these options can enter in contracts concerning the services offered by specialized investment funds.

We have shown that there is no interest in negotiating a covenant which would allow the firm to abandon the investment opportunity if the decision variable is below a specified level at the end of the implementation delay. The so-called Parisian and exponential criterions clearly yield higher values for the investment opportunity than the case when the investment cannot be cancelled. Nevertheless although the exponential abandonment option seems to be closer to the first best abandonment policy, the Parisian investment policy can be probably be implemented in an easier way and at a lower writing cost. Given the recent developments in the literature on incomplete contracts, this aspect may be important since a Parisian criterion can be enforced at a lower cost.

The approach presented in this paper is general enough to allow us to extend the scope of our analysis to other rigidities in the investment process. We focused in this paper on the delay existing between the investment decision and its real implementation. Other imperfections or rigidities can be considered such as noise existing in the information available to the investor concerning the profitability of his investment opportunity or the competition for corporate resources due to the capital allocation process within firms. At the technical level the present model can be extended to take into account entry and exit decisions at a cost of a heavy mathematical treatment. Also, other exotic times can be devised, corresponding to various levels of the freedom with respect to outside parties.

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Chapter 3 THE MATHEMATICS OF DELAYED INVESTMENT DECISION[1]

This chapter presents no fundamentally new result, but proposes a varied range of approaches to the computations involved by stopping times related the Brownian Motion and its excursions. The chapter is organized as follows. The first section is dedicated to the study of a family of simple but nevertheless useful stopping times. The second section defines Parisian stopping times, and provides two proofs for the calculation of their Laplace transform. The first and original proof is due to Chesney, Jeanblanc, Yor (1997). A second proof is proposed, relying on a limit behavior of the stopping times introduced in the second section. In the third section, we introduce a third demonstration of the result on Parisian times, thanks to an extension of a known method, due to Leuridan, and presented as an exercise in Revuz, Yor (1990). The "Parisian" name comes among other things from the numerous meanders of the Seine river in Paris. Another extension of this approach will be used to derive new results under different conditions in the following chapter.

In the whole chapter we will focus on various functionals for a drifted Brownian motion $(Z_t = B_t + bt, t \geq 0)$, where b is a real number and $(B_t, t \geq 0)$ a Brownian motion.

A simple and interesting stopping time

This stopping time's Laplace transform is used to study the case of the "European abandonment option" in the preceding chapter. These stopping times could have been christened the "Maastricht" times, since they are more European than the Parisian ones.

Computation of the first instant where d units of time after having hit a level, the Brownian Motion is above another level

We want to define a stopping time that helps us model delayed investment decision. One way to do this is to consider that if a decision is made at a certain time, then at the end of the delay one may have the choice to either confirm it or cancel it. We consider that the kind of decisions we are interested in depend on the level of some reference variable. This option to cancel the project can be modelled as followed: the investment is made when the reference variable, after the delay, is above some threshold. The event that triggers the count-down of the delay is when the variable has hit some decision level. If we translate all that in Brownian Motion terms, what we want to do is look at the first instant when, d units of time after being at 0, the Brownian Motion is above some level $-a$. If it is not above $-a$, then we have to wait a little until it goes back to zero, and then wait again for d units of time to check whether the process is above $-a$ or not and so on...

Let us define $\nu_d^{-a,0}$, the first instant when, d units of time after reaching 0, the Brownian Motion is above $-a$. To be able to formally define $\nu_d^{-a,0}$, we create the series of random times $T^{(i)}$. $T^{(0)} = d$. If the Brownian Motion is above $-a$ at that

[1] THE PART OF THIS CHAPTER THAT FOCUSES ON THE ALTERNATIVE PROOFS OF THE THEOREM OF CHESNEY, JEANBLANC AND YOR IS TO APPEAR IN THE *ADVANCES IN APPLIED PROBABILITY*, UNDER THE TITLE "PARISIAN OPTIONS: A SIMPLIFIED APPROACH WITHOUT EXCURSIONS"

time, then $\nu_d^{-a,0} = T^{(0)} = d$. If not, we look at the first instant after $T^{(0)}$ when the Brownian Motion reaches 0, that is $T^{T^{(0)},0}$. Then, we write $T^{(1)} = T^{T^{(0)},0} + d$, that is $T^{(1)}$ is d units of time after the Brownian Motion has reached zero. If the Brownian Motion is above $-a$ at $T^{(1)}$, and if it was below $-a$ in $T^{(0)}$ then $\nu_d^{-a,0} = T^{(1)}$. Following the same procedure one can generate the series of $T^{(i)}$. We have $T^{(2)} = T^{T^{(1)},0} + d$ and more generally, $T^{(i+1)} = T^{T^{(i)},0} + d$.

Now we define

$$\begin{aligned}
\widehat{T}^{(0)} &= T^{(0)}\mathbb{I}_{B_{T^{(0)}}\geq -a} \\
\widehat{T}^{(1)} &= T^{(1)}\mathbb{I}_{B_{T^{(1)}}\geq -a}\mathbb{I}_{\widehat{T}^{(0)}=0} \text{ and generally} \\
\widehat{T}^{(i)} &= T^{(i)}\mathbb{I}_{B_{T^{(i)}}\geq -a}\prod_{k=0}^{i-1}\mathbb{I}_{\widehat{T}^{(k)}=0} \\
&= T^{(i)}\mathbb{I}_{B_{T^{(i)}}\geq -a}\prod_{k=0}^{i-1}\mathbb{I}_{B_{T^{(k)}}<-a}
\end{aligned}$$

Finally, we can write

$$\nu_d^{-a,0} = \sum_{i\geq 0}\widehat{T}^{(i)} = \sup_{i\geq 0}\widehat{T}^{(i)} = T^{(i*)} \tag{3.1}$$

with

$$i^* = \sum_{i\geq i} i\mathbb{I}_{B_{T^{(i)}}\geq -a}\prod_{k=0}^{i-1}\mathbb{I}_{B_{T^{(k)}}<-a}.$$

Note that if $\widehat{T}^{(i)} = 0$ for all i then $\nu_d^{-a,0} = 0$ by definition.

The $\widehat{T}^{(i)}$ are not independent. Indeed $\mathbb{P}\left(\widehat{T}^{(1)} \neq 0, \widehat{T}^{(2)} \neq 0\right) = 0$ because of the characteristic function in the expression of $\widehat{T}^{(2)}$, while $\mathbb{P}\left(\widehat{T}^{(1)} \neq 0\right)$ and $\mathbb{P}\left(\widehat{T}^{(1)} \neq 0\right)$ are both strictly positive. However, the times $T^{T^{(i)},0} - T^{(i)}$ are independent thanks to the independence of Brownian increments before and after a stopping time. Also, the times $T^{(i)} - T^{(i-1)} = d + \inf\left\{s : B_{T^{(i)}+s} = 0\right\}$ are independent. Since $T^{T^{(i-1)},0}$ is a stopping time, $\left(B_{T^{T^{(i-1)},0}+s} - B_{T^{T^{(i-1)},0}}, s \geq 0\right)$ is independent from $\mathcal{F}_{T^{T^{(i-1)},0}}$ thanks to the Markov property of the Brownian Motion. Noticing that $B_{T^{T^{(i-1)},0}} = 0$ and $B_{T^{(i)}} = T^{T^{(i-1)},0} + d$, we conclude that $B_{T^{(i)}}$ is independent from $\mathcal{F}_{T^{T^{(i-1)},0}}$ and $B_{T^{(i)}} \overset{d}{=} B_d$.

Lemma 2 $\nu_d^{-a,0}$ *is a stopping time and it is finite.*

Proof. We write for all t

$$\begin{aligned}
\left\{\nu_d^{-a,0} < t\right\} &= \left\{\sup_i \widehat{T}^{(i)} < t\right\} \\
&= \left\{\forall i \geq 1, T^{(i)}\mathbb{I}_{B_{T^{(i)}}\geq -a}\prod_{k=0}^{i-1}\mathbb{I}_{B_{T^{(k)}}<-a} < t\right\}\bigcap\left\{T^{(0)}\mathbb{I}_{B_{T^{(0)}}\geq -a} < t\right\} \\
&= \left(\left\{\exists i : B_{T^{(i)}} \geq -a \text{ and } T^{(i)} < t\right\}\bigcup\left\{\nu_d^{-a,0} = 0\right\}\right) \subset \mathcal{F}_t.
\end{aligned}$$

In the last equality, including the possibility that $\nu_d^{-a,0} = 0$ covers the case when all the $\widehat{T}^{(i)}$ are zero. Therefore $\nu_d^{-a,0}$ is a stopping time. Now, let us consider

$$\begin{aligned}
\left\{\nu_d^{-a,0} = +\infty\right\} &= \bigcap_{i=0}^{+\infty} \left\{B_{T^{(i)}} < -a\right\} \\
\mathbb{P}\left(\nu_d^{-a,0} = +\infty\right) &= \prod_{i=0}^{+\infty} \mathbb{P}\left(B_{T^{(i)}} < -a\right)
\end{aligned}$$

since the $B_{T^{(i)}}$ are independent. In addition, $\mathbb{P}\left(B_{T^{(i)}} < -a\right) = \mathbb{P}\left(B_d < -a\right) < 1$. In consequence, $\mathbb{P}\left(\nu_d^{-a,0} = +\infty\right) = 0$ and $\nu_d^{-a,0}$ is finite. ■

We have then the following result

Proposition 3 *For all positive λ and positive α, $\nu_d^{-a,0}$ and $B_{\nu_d^{-a,0}}$ are independent and*

$$\begin{aligned}
\mathbb{E}\left[e^{-\lambda \nu_d^{-a,0}}\right] &= e^{-\lambda d} \frac{\mathcal{N}\left(\frac{a}{\sqrt{d}}\right)}{\mathcal{N}\left(\sqrt{2\lambda d} + \frac{a}{\sqrt{d}}\right)} \\
\mathbb{E}\left[f\left(B_{\nu_d^{-a,0}}\right)\right] &= \frac{\int_{-a}^{\infty} dz \frac{f(z)}{\sqrt{2\pi d}} e^{-\frac{z^2}{2d}}}{\mathcal{N}\left(\frac{a}{\sqrt{d}}\right)}.
\end{aligned}$$

Proof. First, we show the independence of $B_{\nu_d^{-a,0}}$ and $\nu_d^{-a,0}$, by calculating their joint Laplace Transform:

$$\mathbb{E}\left[e^{-\lambda \nu_d^{-a,0} - \alpha B_{\nu_d^{-a,0}}}\right] = \mathbb{E}\left[\left(\sum_{i=1}^{\infty} e^{-\lambda T^{(i)} - \alpha B_{T^{(i)}}} \mathbb{I}_{B_{T^{(i)}} \geq -a} \prod_{k=0}^{i-1} \mathbb{I}_{B_{T^{(k)}} < -a}\right) + \mathbb{I}_{B_d \geq -a} e^{-\lambda d - \alpha B_d}\right].$$

Now we write $T^{(i)}$ as the following sum of independent terms:

$$T^{(i)} = T^{(0)} + id + \sum_{k=0}^{i-1} \left(T^{T^{(k)},0} - T^{(k)}\right) = (i+1)\, d + \sum_{k=0}^{i-1} \left(T^{T^{(k)},0} - T^{(k)}\right).$$

Replacing in the equation and conditioning gives

$$\begin{aligned}
&\mathbb{E}\left[e^{-\lambda \nu_d^{-a,0} - \alpha B_{\nu_d^{-a,0}}}\right] \\
&= \mathbb{E}\left[\left(\sum_{i=1}^{\infty} e^{-\alpha B_{T^{(i)}}} \mathbb{I}_{B_{T^{(i)}} \geq -a} e^{-\lambda d} \prod_{k=0}^{i-1} \mathbb{I}_{B_{T^{(k)}} < -a} e^{-\lambda d - \lambda\left(T^{T^{(k)},0} - T^{(k)}\right)}\right) + \mathbb{I}_{B_d > -a} e^{-\lambda d - \alpha B_d}\right] \\
&= \sum_{i=1}^{\infty} \mathbb{E}\left[\mathbb{E}\left[e^{-\alpha B_{T^{(i)}}} \mathbb{I}_{B_{T^{(i)}} \geq -a} \middle| \mathcal{F}_{T^{T^{(i-1)},0}}\right] e^{-\lambda d} \prod_{k=0}^{i-1} \mathbb{I}_{B_{T^{(k)}} < -a} e^{-\lambda d - \lambda\left(T^{T^{(k)},0} - T^{(k)}\right)}\right] \\
&\quad + \mathbb{E}\left[\mathbb{I}_{B_d > -a} e^{-\lambda d - \alpha B_d}\right],
\end{aligned}$$

where we used the fact that all $B_{T^{(k)}}$, $T^{(k)}$ and $T^{T^{(k)},0}$ are $\mathcal{F}_{T^{T^{(i-1)},0}}$-measurable for $k \leq i-1$. Besides, we know that $B_{T^{(i)}}$ is independent from $\mathcal{F}_{T^{T^{(i-1)},0}}$ and $B_{T^{(i)}} \overset{d}{=}$

B_d. Now considering the independence of the terms $\left(\mathbb{I}_{B_{T^{(k)}}<-a} e^{-\lambda d-\lambda\left(T^{T^{(k)},0}-T^{(k)}\right)}, k \geq 0\right)$ we can write

$$
\begin{aligned}
& \mathbb{E}\left[e^{-\lambda \nu_d^{-a,0}-\alpha B_{\nu_d^{-a,0}}}\right] \\
= & \mathbb{E}\left[e^{-\alpha B_d} \mathbb{I}_{B_d \geq -a}\right] e^{-\lambda d}\left(1+\sum_{i=1}^{\infty} \mathbb{E}\left[\prod_{k=0}^{i-1} \mathbb{I}_{B_{T^{(k)}}<-a} e^{-\lambda\left(T^{T^{(k)},0}-T^{(k)}\right)}\right]\right) \\
= & \mathbb{E}\left[e^{-\alpha B_d} \mathbb{I}_{B_d \geq -a}\right] e^{-\lambda d}\left(1+\sum_{i=1}^{\infty} \prod_{k=0}^{i-1} \mathbb{E}\left[\mathbb{I}_{B_d \leq -a} e^{-\lambda T^{d,0}}\right]\right) \\
= & \frac{\mathbb{E}\left[e^{-\alpha B_d} \mathbb{I}_{B_d \geq -a}\right] e^{-\lambda d}}{1-\mathbb{E}\left[\mathbb{I}_{B_d \leq -a} e^{-\lambda T^{d,0}}\right]}.
\end{aligned}
$$

So we have

$$
\mathbb{E}\left[e^{-\alpha B_{\nu_d^{-a,0}}}\right]=\frac{\mathbb{E}\left[e^{-\alpha B_d} \mathbb{I}_{B_d \geq -a}\right]}{1-\mathbb{E}\left[\mathbb{I}_{B_d \leq -a}\right]} \text{ and } \mathbb{E}\left[e^{-\lambda \nu_d^{-a,0}}\right]=\frac{e^{-\lambda d} \mathbb{E}\left[\mathbb{I}_{B_d \geq -a}\right]}{1-\mathbb{E}\left[\mathbb{I}_{B_d \leq -a} e^{-\lambda T^{d,0}}\right]},
$$

and the independence of $\nu_d^{-a,0}$ and $B_{\nu_d^{-a,0}}$.

Now, straightforward calculations give

$$
\mathbb{E}\left[\mathbb{I}_{B_d>-a}\right]=\mathcal{N}\left(\frac{a}{\sqrt{d}}\right)
$$

and from Markov property,

$$
\begin{aligned}
\mathbb{E}\left[\mathbb{I}_{B_d \leq -a} e^{-\lambda T^{d,0}}\right] & = e^{-\lambda d} \mathbb{E}\left[\mathbb{I}_{B_d \leq -a} e^{-\left|B_d\right| \sqrt{2 \lambda}}\right] \\
& = \mathcal{N}\left(-\sqrt{2 \lambda d}-\frac{a}{\sqrt{d}}\right).
\end{aligned}
$$

Thus, we obtain

$$
\mathbb{E}\left[e^{-\lambda \nu_d^{-a,0}}\right]=e^{-\lambda d} \frac{\mathcal{N}\left(\frac{a}{\sqrt{d}}\right)}{\mathcal{N}\left(\sqrt{2 \lambda d}+\frac{a}{\sqrt{d}}\right)}.
$$

As for the position of the Brownian Motion, we notice using the Markov property of the process that, at that time, the process is only conditioned by the fact it was at the level 0 d units of time before $\nu_d^{-a,0}$, and that it is higher than $-a$. Therefore we just have to compute

$$
\begin{aligned}
\mathbb{E}\left[f\left(B_{\nu_d^{-a,0}}\right)\right] & = \mathbb{E}\left[f\left(B_d\right) \mid B_d \geq -a\right] \\
& = \frac{\int_{-a}^{\infty} dz \frac{f(z)}{\sqrt{2 \pi d}} e^{-\frac{z^2}{2 d}}}{\mathcal{N}\left(\frac{a}{\sqrt{d}}\right)}
\end{aligned}
$$

to finally obtain the result. ■

Laplace transforms for a drifted Brownian motion

First of all, we naturally extend the definition of $\nu_d^{-a,0}$ to $\nu_d^{h',h}$: the first instant when the process is above h', d units of time after having hit h. If the process is under h' d units of time after hitting h, then we wait until it hits h again, and then look d units of time afterwards to see if the process is above h', and so on...

Proposition 4 *For a positive ρ and for $S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma Z_t\right)$ where Z is a Brownian Motion, we have*

$$\mathbb{E}_h\left[e^{-\rho\nu_d^{h',h}(S)}\right] = \frac{e^{-\rho d}\left(1 - \mathcal{N}\left(b\sqrt{d} + \ln\left(\frac{h'}{h}\right)\left(\sigma\sqrt{d}\right)^{-1}\right)\right)}{1 - \mathcal{N}\left(-\sqrt{(2\rho + b^2)\, d} + \ln\left(\frac{h'}{h}\right)\left(\sigma\sqrt{d}\right)^{-1}\right)}$$

$$\mathbb{E}_h\left[S^\gamma_{\nu_d^{h',h}(S)}\right] = h^\gamma e^{d\gamma\left(\mu - \frac{\sigma^2}{2}(1-\gamma)\right)} \frac{1 - \mathcal{N}\left((b + \gamma\sigma)\sqrt{d} + \ln\left(\frac{h'}{h}\right)\left(\sigma\sqrt{d}\right)^{-1}\right)}{1 - \mathcal{N}\left(-b\sqrt{d} + \ln\left(\frac{h'}{h}\right)\left(\sigma\sqrt{d}\right)^{-1}\right)}.$$

Proof. Writing $S_t = S_0 \exp\left(\sigma Z_t^b\right)$ where $Z_t^b = B_t + bt$ and $b = \frac{1}{\sigma}\left(\mu - \frac{\sigma^2}{2}\right)$, we have

$$\mathbb{E}_h\left[e^{-\rho\nu_d^{h',h}(S)}\right] = \mathbb{E}_0\left[e^{-\rho\nu_d^{l,0}(Z^b)}\right]$$

where we noted $l = \frac{1}{\sigma}\ln\left(\frac{h'}{h}\right)$, and

$$\mathbb{E}_h\left[S^\gamma_{\nu_d^{h',h}(S)}\right] = \mathbb{E}_0\left[h^\gamma \exp\left(\gamma\sigma Z^b_{\nu_d^{l,0}(Z^b)}\right)\right].$$

These expressions, thanks to Girsanov's theorem, give

$$\mathbb{E}_h\left[e^{-\rho\nu_d^{h',h}(S)}\right] = \mathbb{E}_0\left[e^{-\left(\rho + \frac{b^2}{2}\right)\nu_d^{l,0}(Z) + bZ_{\nu_d^{l,0}}}\right]$$

and

$$\mathbb{E}_h\left[S^\gamma_{\nu_d^{h',h}(S)}\right] = \mathbb{E}_0\left[h^\gamma \exp\left((b + \gamma\sigma)\, Z_{\nu_d^{l,0}(Z)} - \frac{b^2}{2}\nu_d^{l,0}(Z)\right)\right]$$

Thanks to the independence between the stopping time and the position of the Brownian Motion at that time, we have

$$\begin{aligned}
\mathbb{E}_h\left[e^{-\rho\nu_d^{h',h}(S)}\right] &= \mathbb{E}_0\left[e^{-\left(\rho + \frac{b^2}{2}\right)\nu_d^{l,0}(Z)}\right]\mathbb{E}_0\left[e^{bZ_{\nu_d^{l,0}}}\right] \\
&= e^{-\rho d}\frac{\left(1 - \mathcal{N}\left(b\sqrt{d} + \frac{l}{\sqrt{d}}\right)\right)}{1 - \mathcal{N}\left(-\sqrt{(2\rho + b^2)\, d} + \frac{l}{\sqrt{d}}\right)}
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}_h\left[S^\gamma_{\nu_d^{h',h}(S)}\right] \\
&= h^\gamma \mathbb{E}_0\left[\exp\left((b + \gamma\sigma)\, Z_{\nu_d^{l,0}(Z)}\right)\right]\mathbb{E}_0\left[\exp\left(-\frac{b^2}{2}\nu_d^{l,0}(Z)\right)\right] \\
&= h^\gamma e^{d\gamma\left(\mu - \frac{\sigma^2}{2}(1-\gamma)\right)}\frac{1 - \mathcal{N}\left((b + \gamma\sigma)\sqrt{d} + \frac{l}{\sqrt{d}}\right)}{1 - \mathcal{N}\left(-b\sqrt{d} + \frac{l}{\sqrt{d}}\right)}
\end{aligned}$$

which is the result. ■

Computation of the first instant when the Brownian Motion spends more than d units of time over a non-constant barrier, while having to go back to zero

In the remaining of this chapter, we count time in a non-cumulative manner. When we say we are interested in the time spent by the Brownian Motion above a level, the counter starts at zero for every new excursion above the level. We will start by defining the following stopping times, for a given measurable negative function y

$$H^{t,y} = \inf\left\{s \geq t : B_s = y\left(s-t\right)\right\}.$$

$H^{t,y}$ is the first instant when after t the Brownian Motion B hits y. Then we define the stopping time $\eta_d^{y,0}$, the first instant when the Brownian Motion spends more than d units of time over y, the countdown starting every time the Brownian Motion reaches 0 after reaching y. An entirely formal definition for $\eta_d^{y,0}$ can be written, in a very similar fashion to the definition of $\nu_d^{-a,0}$. $\eta_d^{y,0}$ is like a Parisian stopping time, but without the issue of determining exactly the last zero of the excursion. Indeed, when we consider an excursion of the Brownian Motion, one important issue is that B vibrates around zero, and therefore there are an infinity of very small excursions that tend to complicate the manipulation of the concept of excursion. Considering the stopping time $\eta_d^{y,0}$ instead, we do not have this problem if we take $y\left(0\right) < 0$; in this case B may vibrate around zero, but it will not jump to $y\left(0\right)$ immediately.

We are interested in a particular y, of the form

$$y\left(t\right) = -\varepsilon - \beta t.$$

We have then the following result

Proposition 5 *For all positive λ and negatively bounded f, $\eta_d^{y,0}$ and $B_{\eta_d^{y,0}}$ are independent and*

$$\begin{aligned}
\mathbb{E}\left[e^{-\lambda\eta_d^{y,0}}\right] &= \frac{e^{-\lambda d}\int_d^\infty dt \frac{\varepsilon\exp\left(-\frac{(\varepsilon+\beta t)^2}{2t}\right)}{\sqrt{2\pi t^3}}}{1-e^{-\varepsilon\sqrt{2\lambda}}\int_0^d dt\frac{\varepsilon\exp\left(-\left(\lambda+\beta\sqrt{2\lambda}\right)t-\frac{(\varepsilon+\beta t)^2}{2t}\right)}{\sqrt{2\pi t^3}}} \\
\mathbb{E}\left[f\left(B_{\nu_d^{-a,0}}\right)\right] &= \frac{\int_{-\infty}^{\infty} dz\frac{f(z)}{\sqrt{2\pi d}}\left(e^{-\frac{z^2}{2d}} - e^{\beta z+\frac{\beta^2 d}{2}-\frac{(\varepsilon+|z+\varepsilon+\beta d|)^2}{2d}}\right)}{\int_d^\infty dt\frac{\varepsilon\exp\left(-\frac{(\varepsilon+\beta t)^2}{2t}\right)}{\sqrt{2\pi t^3}}}.
\end{aligned}$$

Remark 1 *If the barrier is constant, the expression simplifies and can be re-written as:*

$$\mathbb{E}\left[e^{-\lambda\eta_d^{y,0}}\right] = \frac{e^{-\lambda d}\int_d^\infty dt\frac{\varepsilon\exp\left(-\frac{\varepsilon^2}{2t}\right)}{\sqrt{2\pi t^3}}}{1-e^{-2\varepsilon\sqrt{2\lambda}}+e^{-\varepsilon\sqrt{2\lambda}}\int_d^{+\infty} dt\frac{\varepsilon\exp\left(-\lambda t-\frac{\varepsilon^2}{2t}\right)}{\sqrt{2\pi t^3}}}.$$

This expression shows more clearly how $\mathbb{E}\left[e^{-\lambda\eta_d^{y,0}}\right]$ behaves as ε is near zero.

Proof. The proof is based on the Markov property of Brownian Motion, and follows the one developed in the preceding subsection. We first concentrate on the Laplace transform of the law of $\eta_d^{y,0}$. We write directly (the formalization of the reasoning has been done in the previous subsection for $\nu_d^{-a,o}$):

$$\begin{aligned}\mathbb{E}\left[e^{-\lambda\eta_d^{y,0}}\right] &= \sum_{i=0}^{\infty}\left(\mathbb{E}\left[\mathbb{I}_{H^{0,y}\leq d}e^{-\lambda\left(H^{0,y}+H^{H^{0,y},0}\right)}\right]\right)^i\mathbb{E}\left[\mathbb{I}_{H^{0,y}>d}e^{-\lambda d}\right] \\ &= \frac{\mathbb{E}\left[\mathbb{I}_{H^{0,y}>d}e^{-\lambda d}\right]}{1-\mathbb{E}\left[\mathbb{I}_{H^{0,y}\leq d}e^{-\lambda\left(H^{0,y}+H^{H^{0,y},0}\right)}\right]}.\end{aligned}$$

Now, we use the fact that for a Brownian Motion B,

$$\begin{aligned}H^{0,y}(B) &= \inf\{t\geq 0: B_t=-\varepsilon-\beta t\} \\ &= \inf\{t\geq 0: B_t+\beta t=-\varepsilon\} \\ &= T_{-\varepsilon}\left(B^{\beta}\right).\end{aligned}$$

We also know that $B_{H^{0,y}}=-\varepsilon-\beta H^{0,y}$. Also, noticing that the law of $H^{H^{0,y},0}$ only depends on $H^{0,y}$ through the position of the Brownian Motion at that time, we get that in law

$$H^{H^{0,y},0}=T_{-\varepsilon-\beta H^{0,y}}(W)$$

For an independent Brownian Motion W. These considerations allow us to write

$$\begin{aligned}&\mathbb{E}\left[\mathbb{I}_{H^{0,y}\leq d}e^{-\lambda\left(H^{0,y}+H^{H^{0,y},0}\right)}\right] \\ =\ &\mathbb{E}\left[\mathbb{I}_{T_{-\varepsilon}\left(B^{\beta}\right)\leq d}e^{-\lambda T_{-\varepsilon}\left(B^{\beta}\right)-\left(\varepsilon+\beta T_{-\varepsilon}\left(B^{\beta}\right)\right)\sqrt{2\lambda}}\right] \\ =\ &e^{-\varepsilon\sqrt{2\lambda}}\int_0^d dt\frac{\varepsilon\exp\left(-\left(\lambda+\beta\sqrt{2\lambda}\right)t-\frac{(\varepsilon+\beta t)^2}{2t}\right)}{\sqrt{2\pi t^3}}.\end{aligned}$$

On the other hand we get easily

$$\mathbb{E}\left[\mathbb{I}_{H^{0,y}>d}e^{-\lambda d}\right]=e^{-\lambda d}\int_d^{\infty}dt\frac{\varepsilon\exp\left(-\frac{(\varepsilon+\beta t)^2}{2t}\right)}{\sqrt{2\pi t^3}}.$$

This last results completes the computation, and we have

$$\mathbb{E}\left[e^{-\lambda\eta_d^{y,0}}\right]=\frac{e^{-\lambda d}\int_d^{\infty}dt\frac{\varepsilon\exp\left(-\frac{(\varepsilon+\beta t)^2}{2t}\right)}{\sqrt{2\pi t^3}}}{1-e^{-\varepsilon\sqrt{2\lambda}}\int_0^d dt\frac{\varepsilon\exp\left(-\left(\lambda+\beta\sqrt{2\lambda}\right)t-\frac{(\varepsilon+\beta t)^2}{2t}\right)}{\sqrt{2\pi t^3}}}.$$

As for the position of the Brownian Motion, we first notice, still using the Markov property of the process that, at that time, the process is only conditioned by the fact it was at the level 0 d units of time before $\eta_d^{y,0}$, and that it has not hit the barrier before d. We have

$$\begin{aligned}\mathbb{E}\left[f\left(B_{\eta_d^{y,0}}\right)\right] &= \mathbb{E}\left[f(B_d)\middle|\,T_{-\varepsilon}\left(B^{\beta}\right)\geq d\right] \\ &= \mathbb{E}\left[f\left(B_d^{\beta}-\beta d\right)\middle|\,T_{-\varepsilon}\left(B^{\beta}\right)\geq d\right] \\ &= \frac{\mathbb{E}\left[f\left(B_d^{\beta}-\beta d\right)\mathbb{I}_{\inf_{u\leq d}B_u^{\beta}\geq-\varepsilon}\right]}{\mathbb{P}\left[T_{-\varepsilon}\left(B^{\beta}\right)\geq d\right]}\end{aligned}$$

Using the well-known joint law between the Brownian Motion and its minimum (cf. Borodin and Salminen (1996), formula 1.2.8, p. 199), we get

$$\begin{aligned}\mathbb{E}\left[f\left(B_{\eta_d^{y,0}}\right)\right] &= \frac{\mathbb{E}\left[f\left(B_d^{\beta}-\beta d\right)\left(1-\mathbb{I}_{\inf_{u\le d}B_u^{\beta}\le-\varepsilon}\right)\right]}{\int_d^{\infty}dt\frac{\varepsilon\exp\left(-\frac{(\varepsilon+\beta t)^2}{2t}\right)}{\sqrt{2\pi t^3}}} \\ &= \frac{\int_{-\infty}^{\infty}dz\frac{f(z)e^{-\frac{z^2}{2d}}}{\sqrt{2\pi d}}-\int_{-\infty}^{\infty}dz\frac{f(z-\beta d)e^{\beta z-\frac{\beta^2 d}{2}-\frac{(\varepsilon+|z+\varepsilon|)^2}{2d}}}{\sqrt{2\pi d}}}{\int_d^{\infty}dt\frac{\varepsilon\exp\left(-\frac{(\varepsilon+\beta t)^2}{2t}\right)}{\sqrt{2\pi t^3}}} \\ &= \frac{\int_{-\infty}^{\infty}dz\frac{f(z)}{\sqrt{2\pi d}}\left(e^{-\frac{z^2}{2d}}-e^{\beta z+\frac{\beta^2 d}{2}-\frac{(\varepsilon+|z+\varepsilon+\beta d|)^2}{2d}}\right)}{\int_d^{\infty}dt\frac{\varepsilon\exp\left(-\frac{(\varepsilon+\beta t)^2}{2t}\right)}{\sqrt{2\pi t^3}}},\end{aligned}$$

and this is the announced result. ■

Laplace transforms for the drifted Brownian motion

Now, we are interested in similar results for the geometric Brownian Motion. We have the following

Proposition 6 *For a positive ρ and for $S_t = S_0\exp\left(\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma Z_t\right)$ where Z is a Brownian Motion, and $g(t) = he^{-\sigma(\varepsilon+\beta t)}$ we have*

$$\mathbb{E}_h\left[e^{-\rho\eta_d^{g,h}(S)}\right] = \mathbb{E}_0\left[e^{-\left(\rho+\frac{b^2}{2}\right)\eta_d^{y,0}(Z)}\right]\mathbb{E}_0\left[e^{bZ_{\eta_d^{y,0}}}\right]$$

and

$$\mathbb{E}_h\left[S^{\gamma}_{\eta_d^{g,h}(S)}\right] = h^{\gamma}\mathbb{E}_0\left[\exp\left((b+\gamma\sigma)Z_{\eta_d^{y,0}(Z)}\right)\right]\mathbb{E}_0\left[\exp\left(-\frac{b^2}{2}\eta_d^{y,0}(Z)\right)\right].$$

Proof. Recalling that we have $S_t = S_0\exp\left(\sigma Z_t^b\right)$ we write

$$\mathbb{E}_h\left[e^{-\rho\eta_d^{g,h}(S)}\right] = \mathbb{E}_0\left[e^{-\rho\eta_d^{y,0}(Z^b)}\right]$$

and

$$\mathbb{E}_h\left[S^{\gamma}_{\eta_d^{g,h}(S)}\right] = \mathbb{E}_0\left[h^{\gamma}\exp\left(\gamma\sigma Z^b_{\eta_d^{y,0}(Z^b)}\right)\right].$$

These expressions, thanks to Girsanov's theorem, give

$$\mathbb{E}_h\left[e^{-\rho\eta_d^{g,h}(S)}\right] = \mathbb{E}_0\left[e^{-\left(\rho+\frac{b^2}{2}\right)\eta_d^{y,0}(Z)+bZ_{\eta_d^{y,0}(Z)}}\right]$$

and

$$\mathbb{E}_h\left[S^{\gamma}_{\eta_d^{g,h}(S)}\right] = \mathbb{E}_0\left[h^{\gamma}\exp\left((b+\gamma\sigma)Z_{\eta_d^{y,0}(Z)}-\frac{b^2}{2}\eta_d^{y,0}(Z)\right)\right].$$

Thanks to the independence between the stopping time and the position of the Brownian Motion at that time, we have

$$\mathbb{E}_h\left[e^{-\rho\eta_d^{g,h}(S)}\right] = \mathbb{E}_0\left[e^{-\left(\rho+\frac{b^2}{2}\right)\eta_d^{y,0}(Z)}\right]\mathbb{E}_0\left[e^{bZ_{\eta_d^{y,0}}}\right]$$

which can be explicitly computed, and

$$\mathbb{E}_h\left[S^{\gamma}_{\eta_d^{g,h}(S)}\right] = h^{\gamma}\mathbb{E}_0\left[\exp\left((b+\gamma\sigma)\,Z_{\eta_d^{y,0}(Z)}\right)\right]\mathbb{E}_0\left[\exp\left(-\frac{b^2}{2}\eta_d^{y,0}(Z)\right)\right]$$

as said in the proposition. ■

The Parisian stopping time

In this case, the manager invests at $T_h + d$ only if the decision variable remains above the optimal investment threshold during the implementation delay. The decision triggering criterium is a so-called Parisian stopping time depending on the size of excursions of the state variable over the optimal investment threshold. In this paragraph the size of the excursion is equal to the implementation delay d. This is different from the previous case since here every excursion above zero is accounted for, however small it may be. Previously, the process had to go down to $-\varepsilon$ to stop the count-down.

First of all, we define the following random variables, all linked to a random process X:

$$\begin{aligned} g_t^a(X) &= \sup\{s \leq t : X_s = a\} \\ H_{a,d}^+(X) &= \inf\{t \geq 0 : (t - g_t^a(X)) \geq d, X_t \geq a\}. \end{aligned}$$

Thus $g_t^a(X)$ represents the last time the process X crossed the level a. It can be checked it is not a stopping time for the Brownian filtration $\mathcal{F}_t$, but for the slow filtration $\mathcal{G} = (\sigma(sgn(B_t)) \vee \mathcal{F}_{g_t})_{t\geq 0}$ which represents the information on the Brownian motion until its last zero plus the knowledge of its sign after this. $H_{a,d}^+(X)$ is therefore the first instant when the process has spent d units of time consecutively over the level a. We define a Parisian investment decision criterion very simply as $H_{h,d}^+(S)$. We write H_d^+ for $H_{0,d}^+$.

The Laplace transform of the first time a positive Brownian excursion reaches a certain length

Theorem 7 (Chesney Jeanblanc Yor 1997) *If B is a Brownian motion starting from zero, then*

$$\forall\lambda \geq 0,\ \mathbb{E}\left[\exp\left(-\lambda H_d^+(B)\right)\right] = \frac{1}{\Phi\left(\sqrt{2\lambda d}\right)}$$

if $\Phi(x) = \int_0^{+\infty} z\exp\left(zx - \frac{z^2}{2}\right)dz = 1 + \sqrt{2\pi}xe^{-\frac{x^2}{2}}\mathcal{N}(x)$.

The original proof is given in Chesney, Jeanblanc and Yor (1997). In subsequent sections, we will prove the result

- by taking to the limit the results developed in the preceding section
- with the use of excursion theory and the description of Itô's measure.

The theorem can be easily extended to a drifted Brownian Motion:

Theorem 8 *If $Z_t = B_t + bt$, the Laplace transform of $H_d^+(Z)$ is given by*

$$\forall\lambda \geq 0,\ \mathbb{E}\left[\exp\left(-\lambda H_d^+(Z)\right)\right] = \frac{\Phi\left(b\sqrt{d}\right)}{\Phi\left(\sqrt{(2\lambda + b^2)\,d}\right)}.$$

Proof. Our goal is to compute

$$\mathbb{E}\left[\exp\left(-\lambda H_d^+(Z)\right)\right]$$

Using Girsanov's theorem, we write for $\lambda \geq 0$

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\lambda H_d^+(Z)\right)\right] &= \mathbb{E}_{\mathbb{L}}\left[\exp\left(-\lambda H_d^+(Z)\right) \left.\frac{d\mathbb{P}}{d\mathbb{L}}\right| \quad \mathcal{F}_{H_d^+(Z)}\right] \\
&= \mathbb{E}_{\mathbb{P}}\left[\exp\left(-\left(\lambda + \frac{b^2}{2}\right) H_d^+(B)\right) \exp\left(bB_{H_d^+(B)}\right)\right]
\end{aligned}$$

But we know that H_d^+ and $B_{H_d^+}$ are independent, and in law, $B_{H_d^+} = m_d = \sqrt{d}m_1$. Therefore

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\lambda H_d^+(Z)\right)\right] = \mathbb{E}_{\mathbb{P}}\left[e^{\left(-\left(\lambda + \frac{b^2}{2}\right) H_d^+(B)\right)}\right] \int_0^{+\infty} e^{bx} \frac{x}{d} \exp\left(-\frac{x^2}{2d}\right) dx.$$

After some simplifications, we get for any $\lambda \geq 0$

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\lambda H_d^+(Z)\right)\right] &= \frac{1}{\Phi\left(\sqrt{(2\lambda + b^2)\, d}\right)} \int_0^{+\infty} e^{bx} \frac{x}{d} \exp\left(-\frac{x^2}{2d}\right) dx \\
&= \frac{\Phi\left(b\sqrt{d}\right)}{\Phi\left(\sqrt{(2\lambda + b^2)\, d}\right)}.
\end{aligned}$$

as expected. ■

Theorem 9 *If $Z_t = B_t + bt$, we have the following Laplace transform*

$$\forall \lambda \geq 0,\ \mathbb{E}_{\mathbb{P}}\left[\exp\left(-\lambda H_{a,d}^+(Z)\right)\right] = \exp\left(ba - |a| \sqrt{2\lambda + b^2}\right) \frac{\Phi\left(b\sqrt{d}\right)}{\Phi\left(\sqrt{(2\lambda + b^2)\, d}\right)}.$$

Proof. We write that

$$H_{a,d}^+(Z) = T_a(Z) + H_d^+\left(Z \circ \theta_{T_a(Z)} - a\right)$$

where θ is the so-called "shift operator" on the canonical space Ω. This means that, using the strong Markov property and the fact that the increments after and before T_a are independent, we can write

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\lambda H_{a,d}^+(Z)\right)\right] &= \mathbb{E}_{\mathbb{P}}\left[\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\lambda H_{a,d}^+(Z)\right) \middle| \mathcal{F}_{T_a(Z)}\right]\right] \\
&= \mathbb{E}_{\mathbb{P}}\left[\mathbb{E}_{\mathbb{P}}\left[\exp\left(-\lambda\left(T_a(Z) + H_d^+\left(Z \circ \theta_{T_a(Z)} - a\right)\right)\right) \middle| T_a(Z)\right]\right] \\
&= \mathbb{E}_{\mathbb{P}}\left[\exp\left(-\lambda T_a(Z)\right)\right] \mathbb{E}_{\mathbb{P}}\left[\exp\left(-\lambda H_d^+(Z)\right)\right].
\end{aligned}$$

and we obtain the result. ■

A second proof of Chesney, Jeanblanc, Yor's theorem
Let us start with the following

Lemma 10 *We have* $\lim_{\varepsilon\to 0}\eta_d^{-\varepsilon,0} = H_{0,d}^{+}$ *almost surely.*

Proof. First, let us compare $\eta_d^{-\varepsilon,0}$ and $H_{0,d}^{+}$. For any continuous path starting from zero, we have $\eta_d^{-\varepsilon,0} \leq H_{0,d}^{+}$. Indeed, any trajectory that spends consecutively d units of time above zero will have a fortiori spent d units of time above $-\varepsilon$. Now, let us introduce $H_{-\varepsilon,d}^{+}$ the Parisian time defined relative to $-\varepsilon$ instead of zero. We always have (on continuous trajectories) $H_{-\varepsilon,d}^{+} \leq \eta_d^{-\varepsilon,0}$. Indeed, the Parisian criterion will be satisfied earlier than the other stopping time because it includes the time spent between $-\varepsilon$ and 0 in the count-down. So, almost surely $H_{-\varepsilon,d}^{+} \leq \eta_d^{-\varepsilon,0} \leq H_{0,d}^{+}$. Now, $\tau_d^{-\varepsilon}$ converges towards $H_{0,d}^{+}$ (just by taking $\varepsilon = 0$). As a consequence, $\eta_d^{-\varepsilon,0}$ converges to $H_{0,d}^{+}$ almost surely. ■

Now, thanks to the above Lemma and to Proposition 5, we can obtain another proof of Theorem 7. Indeed, we have

$$\mathbb{E}\left[e^{-\lambda\eta_d^{y,0}}\right] = \frac{e^{-\lambda d}\int_d^\infty dt\frac{\varepsilon\exp\left(-\frac{(\varepsilon+\beta t)^2}{2t}\right)}{\sqrt{2\pi t^3}}}{1-e^{-\varepsilon\sqrt{2\lambda}}\int_0^d dt\frac{\varepsilon\exp\left(-\left(\lambda+\beta\sqrt{2\lambda}\right)t-\frac{(\varepsilon+\beta t)^2}{2t}\right)}{\sqrt{2\pi t^3}}}.$$

Now, if the function y is simply defined as $y=\varepsilon$, we obtain (adapting the notation)

$$\mathbb{E}\left[e^{-\lambda\eta_d^{\varepsilon,0}}\right] = \frac{e^{-\lambda d}\int_d^\infty dt\frac{\varepsilon\exp\left(-\frac{\varepsilon^2}{2t}\right)}{\sqrt{2\pi t^3}}}{1-e^{-\varepsilon\sqrt{2\lambda}}\int_0^d dt\frac{\varepsilon\exp\left(-\lambda t-\frac{\varepsilon^2}{2t}\right)}{\sqrt{2\pi t^3}}}.$$

where this Laplace transform is that of the first instant where, starting from zero, the Brownian Motion spends d units of time consecutively over $-\varepsilon$. Also, we know that almost surely $\eta_d^{\varepsilon,0} \underset{\varepsilon\to 0}{\to} H_d^{+}$ where H_d^{+} is the Parisian time. So the convergence result applies to the Laplace transform.

Therefore

$$\begin{aligned}
\mathbb{E}\left[e^{-\lambda H_d^{+}}\right] &= \lim_{\varepsilon\to 0}\mathbb{E}\left[e^{-\lambda\eta_d^{\varepsilon,0}}\right] \\
&= \lim_{\varepsilon\to 0}\frac{e^{-\lambda d}\int_d^\infty dt\frac{\varepsilon\exp\left(-\frac{\varepsilon^2}{2t}\right)}{\sqrt{2\pi t^3}}}{1-e^{-\varepsilon\sqrt{2\lambda}}\int_0^d dt\frac{\varepsilon\exp\left(-\lambda t-\frac{\varepsilon^2}{2t}\right)}{\sqrt{2\pi t^3}}} \\
&= \lim_{\varepsilon\to 0}\frac{e^{-\lambda d}\int_d^\infty dt\frac{\varepsilon\exp\left(-\frac{\varepsilon^2}{2t}\right)}{\sqrt{2\pi t^3}}}{1-e^{-2\varepsilon\sqrt{2\lambda}}+e^{-\varepsilon\sqrt{2\lambda}}\int_d^{+\infty} dt\frac{\varepsilon\exp\left(-\lambda t-\frac{\varepsilon^2}{2t}\right)}{\sqrt{2\pi t^3}}}.
\end{aligned}$$

We used the fact that

$$\int_0^\infty dt\frac{\varepsilon\exp\left(-\lambda t-\frac{\varepsilon^2}{2t}\right)}{\sqrt{2\pi t^3}} = \mathbb{E}\left[e^{-\lambda T_\varepsilon}\right] = e^{-\varepsilon\sqrt{2\lambda}},$$

and therefore

$$\int_0^d dt \frac{\varepsilon \exp\left(-\lambda t - \frac{\varepsilon^2}{2t}\right)}{\sqrt{2\pi t^3}} = e^{-\varepsilon\sqrt{2\lambda}} - \int_d^{+\infty} dt \frac{\varepsilon \exp\left(-\lambda t - \frac{\varepsilon^2}{2t}\right)}{\sqrt{2\pi t^3}}.$$

So as to compute the limit in question, the most natural way is to write limited developments. We obtain that the limit is equal to

$$\frac{\sqrt{\frac{2}{\pi d}} e^{-\lambda d}}{2\sqrt{2\lambda} + \int_d^\infty dt \frac{\exp(-\lambda t)}{\sqrt{2\pi t^3}}}$$

However,

$$\begin{aligned} \int_d^\infty dt \frac{\exp(-\lambda t)}{\sqrt{2\pi t^3}} &= 2\sqrt{2\lambda} \int_{\sqrt{2\lambda d}}^\infty dz \frac{\exp\left(-\frac{z^2}{2}\right)}{z^2\sqrt{2\pi}} \text{ by change of variable} \\ &= \frac{e^{-\lambda d}}{\sqrt{2\pi}} \frac{2\sqrt{2\lambda}}{\sqrt{2\lambda d}} - 2\sqrt{2\lambda} \int_{\sqrt{2\lambda d}}^\infty \frac{dz e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \\ &\quad \text{by integration by parts.} \end{aligned}$$

Consequently, we have

$$\begin{aligned} &\lim_{\varepsilon\to 0} \mathbb{E}\left[e^{-\lambda \eta_d^{\varepsilon,0}}\right] \\ &= \frac{\sqrt{\frac{2}{\pi d}} e^{-\lambda d}}{2\sqrt{2\lambda} + \frac{e^{-\lambda d}}{\sqrt{2\pi}} \frac{2\sqrt{2\lambda}}{\sqrt{2\lambda d}} - 2\sqrt{2\lambda} \int_{\sqrt{2\lambda d}}^\infty \frac{dz e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}} \\ &= \frac{\sqrt{\frac{2}{\pi d}} e^{-\lambda d}}{2\sqrt{2\lambda} + \frac{e^{-\lambda d}}{\sqrt{2\pi}} \frac{2\sqrt{2\lambda}}{\sqrt{2\lambda d}} - 2\sqrt{2\lambda}\left(1 - \mathcal{N}\left(\sqrt{2\lambda d}\right)\right)} \\ &= \frac{1}{1 + \sqrt{2\pi d\lambda} e^{-\lambda d} \mathcal{N}\left(\sqrt{2\lambda d}\right)} \\ &= \frac{1}{\phi\left(\sqrt{2\lambda d}\right)} \end{aligned}$$

and this is exactly the result of Chesney, Jeanblanc and Yor's theorem.

We can also apply the same approach to find back the law of $B_{H_d^+}$. We have a similar convergence as above:

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[f\left(B_{\eta_d^{\varepsilon,0}}\right)\right] = \mathbb{E}\left[f\left(B_{H_d^+}\right)\right].$$

We write, using limited developments, that

$$\begin{aligned} &\lim_{\varepsilon\to 0} \mathbb{E}\left[f\left(B_{\eta_d^{\varepsilon,0}}\right)\right] \\ &= \lim_{\varepsilon\to 0} \frac{\int_{-\infty}^\infty dz \frac{f(z)}{\sqrt{2\pi d}} \left(e^{-\frac{z^2}{2d}} - e^{-\frac{(\varepsilon + |z+\varepsilon|)^2}{2d}}\right)}{\int_d^\infty dt \frac{\varepsilon \exp\left(-\frac{\varepsilon^2}{2t}\right)}{\sqrt{2\pi t^3}}} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{\varepsilon\to 0}\frac{\int_{-\varepsilon}^{\infty}dz\frac{f(z)}{\sqrt{2\pi d}}\left(e^{-\frac{z^2}{2d}}-e^{-\frac{(2\varepsilon+z)^2}{2d}}\right)}{\int_d^{\infty}dt\frac{\varepsilon\exp\left(-\frac{\varepsilon^2}{2t}\right)}{\sqrt{2\pi t^3}}}\\
&= \lim_{\varepsilon\to 0}\frac{\int_{-\varepsilon}^{\infty}dz\frac{f(z)}{\sqrt{2\pi d}}\frac{(4\varepsilon+2z)}{d}e^{-\frac{z^2}{2d}}}{\varepsilon\int_d^{\infty}dt\frac{1}{\sqrt{2\pi t^3}}}\\
&= \frac{\frac{\sqrt{2}}{d\sqrt{\pi d}}\int_0^{\infty}dzf\left(z\right)ze^{-\frac{z^2}{2d}}}{\int_d^{\infty}dt\frac{1}{\sqrt{2\pi t^3}}}=\frac{1}{d}\int_0^{\infty}dzf\left(z\right)ze^{-\frac{z^2}{2d}}\\
&= \mathbb{E}\left[f\left(m_1\sqrt{d}\right)\right]
\end{aligned}
$$

where m is the Meander. This is indeed the result from Chesney, Jeanblanc and Yor.

Excursion theory related stopping times

In this section we propose an extension of a known method to calculate "Parisian-style" stopping times Laplace transforms.

Let us study something of the form

$$\int_0^{\infty}dte^{-\rho t}\mathbb{E}\left[f\left(W_t\right)\mathbb{I}_{H_D^+>t}\right].$$

So as to write explicitly this integral, we will use a particular approach of excursions lengths due to Leuridan. Our method will be based on his computation of a Laplace Transform of the law of the longest Brownian excursion up to a time t, which is given as exercise 4.10 in Revuz Yor (1991).

Reformulation of the integral

First of all, we recall the definition of the inverse process of the local time:

$$\tau_s=\inf\left\{t\geq 0:L_t=s\right\}.$$

The points of increase of the local time are the zeros of the Brownian Motion, and therefore the jumps in the inverse process intervene at the extremities of excursions. Let us also define the longest positive excursion up to a time τ_{s-} as

$$l^+\left(\tau_{s-}\right)=\sup\left\{l\geq 0:\exists u,u<s,\left(\tau_u-\tau_{u-}\right)=l,\mathbf{e}_u\geq 0\right\}$$

where $\mathbf{e}$ is the excursion process. This random variable can also be defined up to the last zero as

$$l^+\left(g_t\right)=\sup\left\{\left(d_s-g_s\right):g_t>s\geq 0,W_s\geq 0\right\}.$$

V will be used to denote the length of an excursion, that is $V\left(\mathbf{e}_s\right)=\tau_s-\tau_{s-}$

Now, let us notice we have the following equalities of events

$$
\begin{aligned}
\left(H_D^+>t\right) &= \left(l^+\left(g_t\right)<D\right)\\
&\bigcap\left(\left(W_t\leq 0\right)\bigcup\left(\left(W_t>0\right)\bigcap\left(t-g_t<D\right)\right)\right).
\end{aligned}
$$

We deduce that we can write

$$
\begin{aligned}
&\int_0^{\infty}dte^{-\rho t}\mathbb{E}\left[f\left(W_t\right)\mathbb{I}_{H_D^+>t}\right]\\
&= \mathbb{E}\int_0^{\infty}dte^{-\rho t}f\left(W_t\right)\mathbb{I}_{l^+(g_t)<D}\left(\mathbb{I}_{W_t>0}\mathbb{I}_{t-g_t<D}+\mathbb{I}_{W_t\leq 0}\right).
\end{aligned}
$$

A useful path decomposition based on the balayage principle

We will use the following decomposition formula, but also we will detail a part of its proof, which happens to be sufficient for our needs:

$$\int_0^\infty dt W^t = \int_0^\infty ds W^{\tau_s} \circ \int_0^\infty dt \mathbf{n}^t (\cdot; t < V)$$

where W^t is the law of the Brownian Motion up to t and $\mathbf{n}$ is Itô's measure. This result can be found in Yor (1997).

Let us consider the following path integral, for two positive measurable functionals F_1 and F_2

$$\mathbb{E}\left[\int_0^\infty dt e^{-\rho t} F_1 (B_u, u \leq g_t) F_2 (B_u, g_t \leq u \leq t)\right].$$

We rewrite it and apply a change of variable, namely $t = v + \tau_{s-}$, and obtain

$$\begin{aligned} & \mathbb{E}\left[\sum_s \int_{\tau_{s-}}^{\tau_s} dt e^{-\rho t} F_1 (B_u, u \leq g_t) F_2 (B_u, g_t \leq u \leq t)\right] \\ = \; & \mathbb{E}\left[\sum_s \int_0^{V(\mathbf{e}_s)} dv e^{-\rho(v+\tau_{s-})} F_1 (B_u, u \leq \tau_{s-}) F_2 (\mathbf{e}_s (u), u \leq v)\right]. \end{aligned}$$

Now the first Master formula (see Yor (1997)), also called the Compensation Formula, gives the expected path decomposition:

$$\begin{aligned} & \mathbb{E}\left[\int_0^\infty ds \int \mathbf{n}(d\varepsilon) \int_0^{V(\varepsilon)} dv e^{-\rho(v+\tau_{s-})} F_1 (B_u, u \leq \tau_{s-}) F_2 (\varepsilon (u), u \leq v)\right] \\ = \; & \mathbb{E}\left[\int_0^\infty ds e^{-\rho\tau_{s-}} F_1 (B_u, u \leq \tau_{s-})\right] \int \mathbf{n}(d\varepsilon) \int_0^{V(\varepsilon)} dv e^{-\rho v} F_2 (\varepsilon_u, u \leq v). \end{aligned}$$

Applying this result to the particular case we are studying entails immediately

$$\begin{aligned} & \mathbb{E}\int_0^\infty dt e^{-\rho t} \mathbb{I}_{l^+(g_t)<D} \left(\mathbb{I}_{W_t>0}\mathbb{I}_{t-g_t<D} + \mathbb{I}_{W_t\leq 0}\right) f(W_t) \\ = \; & \mathbb{E}\left[\int_0^\infty ds e^{-\rho\tau_{s-}} \mathbb{I}_{l^+(\tau_{s-})<D}\right] \int \mathbf{n}(d\varepsilon) \int_0^{V(\varepsilon)} dv e^{-\rho v} f(\varepsilon_v) \left(\mathbb{I}_{\varepsilon_v>0}\mathbb{I}_{v<D} + \mathbb{I}_{\varepsilon_v\leq 0}\right). \end{aligned}$$

The second part of the RHS product is known, thanks to Itô's description of the measure $\mathbf{n}$. Indeed, we have

- $\mathbf{n}(V(\varepsilon) \in dv) = \frac{dv}{\sqrt{2\pi v^3}}$
- $\mathbf{n}(\varepsilon \in df | V(\varepsilon) = v) = \mathbb{R}^{(3)}_{0\to 0,v}(\varepsilon \in df)$ where $\mathbb{R}^{(3)}_{0\to 0,v}$ is the law of a Bessel-3 bridge from 0 to 0 of length v.

Computations of the path integral related to the longest excursion

We are interested in $\mathbb{E}\left[\int_0^\infty ds e^{-\rho\tau_{s-}} \mathbb{I}_{l^+(\tau_s)<D}\right]$. We start by noticing that

$$e^{-\rho\tau_{s-}} \mathbb{I}_{l^+(\tau_{s-})<D} = \sum_{0\leq u<s} \left(e^{-\rho\tau_u} \mathbb{I}_{l^+(\tau_u)<D} - e^{-\rho\tau_{u-}} \mathbb{I}_{l^+(\tau_{u-})<D}\right) + 1.$$

But we also write easily that

$$\begin{aligned}\left(l^{+}(\tau_u)<D\right) &= \left(l^{+}(\tau_{u-})<D\right)\\ &\bigcap\left((\mathbf{e}_u\leq 0)\bigcup\left((\mathbf{e}_u>0)\bigcap(\tau_u-\tau_{u-}<D)\right)\right).\end{aligned}$$

In other words, for the longest excursion up to τ_u to be shorter than D, we need the longest of all excursions before the one straddling τ_u to be shorter than D (that is $l^{+}(\tau_{u-})<D$), and either the next one is negative, or it is shorter than D. Therefore, we have

$$\begin{aligned}&e^{-\rho\tau_{s-}}\mathbb{I}_{l^{+}(\tau_{s-})<D}\\ =&\sum_{0\leq u<s}\left(\mathbb{I}_{l^{+}(\tau_{u-})<D}\left(e^{-\rho\tau_u}\left(\mathbb{I}_{\mathbf{e}_u\leq 0}+\mathbb{I}_{\mathbf{e}_u>0}\mathbb{I}_{\tau_u-\tau_{u-}<D}\right)-e^{-\rho\tau_{u-}}\right)\right)+1\\ =&\sum_{0\leq u<s}\left(e^{-\rho\tau_{u-}}\mathbb{I}_{l^{+}(\tau_{u-})<D}\left(e^{-\rho V(\mathbf{e}_s)}\left(\mathbb{I}_{\mathbf{e}_u\leq 0}+\mathbb{I}_{\mathbf{e}_u>0}\mathbb{I}_{\tau_u-\tau_{u-}<D}\right)-1\right)\right)+1.\end{aligned}$$

Taking the expectation and applying the Compensation Formula yields

$$\begin{aligned}&\mathbb{E}\left[e^{-\rho\tau_{s-}}\mathbb{I}_{l^{+}(\tau_{s-})<D}\right]\\ =&\ 1+\mathbb{E}\left[\int_0^s due^{-\rho\tau_{u-}}\mathbb{I}_{l^{+}(\tau_{u-})<D}\right]\int\mathbf{n}(d\varepsilon)\left(e^{-\rho V(\varepsilon)}\left(\mathbb{I}_{\varepsilon\leq 0}+\mathbb{I}_{\varepsilon>0}\mathbb{I}_{V(\varepsilon)<D}\right)-1\right).\end{aligned}$$

If we define

$$\varphi_D(s)=\mathbb{E}\left[e^{-\rho\tau_{s-}}\mathbb{I}_{l^{+}(\tau_{s-})<D}\right]$$

then we know that

$$\mathbb{E}\left[\int_0^\infty dse^{-\rho\tau_{s-}}\mathbb{I}_{l^{+}(\tau_{s-})<D}\right]=\int_0^\infty ds\varphi_D(s)$$

and

$$\varphi_D(s)=1+\int\mathbf{n}(d\varepsilon)\left(e^{-\rho V(\varepsilon)}\left(\mathbb{I}_{\varepsilon\leq 0}+\mathbb{I}_{\varepsilon>0}\mathbb{I}_{V(\varepsilon)<D}\right)-1\right)\int_0^s du\varphi_D(u).$$

Solving the differential equation gives directly

$$\mathbb{E}\left[\int_0^\infty dse^{-\rho\tau_{s-}}\mathbb{I}_{l^{+}(\tau_{s-})<D}\right]=\frac{1}{\int\mathbf{n}(d\varepsilon)\left(1-e^{-\rho V(\varepsilon)}\left(\mathbb{I}_{\varepsilon\leq 0}+\mathbb{I}_{\varepsilon>0}\mathbb{I}_{V(\varepsilon)<D}\right)\right)}.$$

Therefore, we conclude that

$$\int_0^\infty dte^{-\rho t}\mathbb{E}\left[f(W_t)\mathbb{I}_{H_D^{+}>t}\right]=\frac{\int\mathbf{n}(d\varepsilon)\int_0^{V(\varepsilon)}dve^{-\rho v}f(\varepsilon_v)\left(\mathbb{I}_{\varepsilon_v>0}\mathbb{I}_{v<D}+\mathbb{I}_{\varepsilon_v\leq 0}\right)}{\int\mathbf{n}(d\varepsilon)\left(1-e^{-\rho V(\varepsilon)}\left(\mathbb{I}_{\varepsilon\leq 0}+\mathbb{I}_{\varepsilon>0}\mathbb{I}_{V(\varepsilon)<D}\right)\right)}.$$

Using the description of Itô's measure, we can write

$$\begin{aligned}\int_0^\infty dte^{-\rho t}\mathbb{E}\left[f(W_t)\mathbb{I}_{H_D^{+}>t}\right] &= \frac{\frac{1}{2}\int_0^\infty\frac{dv}{\sqrt{2\pi v^3}}\int_0^v due^{-\rho u}\int_0^\infty dr\left(f(r)\mathbb{I}_{u<D}+f(-r)\right)q(r;u,v)}{\int\mathbf{n}(d\varepsilon)\left(\mathbb{I}_{\varepsilon\leq 0}-e^{-\rho V(\varepsilon)}\mathbb{I}_{\varepsilon\leq 0}+\mathbb{I}_{\varepsilon>0}-e^{-\rho V(\varepsilon)}\mathbb{I}_{\varepsilon>0}\mathbb{I}_{V(\varepsilon)<D}\right)}\\ &= \frac{\int_0^\infty\frac{dv}{\sqrt{2\pi v^3}}\int_0^v due^{-\rho u}\int_0^\infty dr\left(f(r)\mathbb{I}_{u<D}+f(-r)\right)q(r;u,v)}{\int_0^\infty\frac{dv}{\sqrt{2\pi v^3}}\left(1-e^{-\rho v}\right)+\int_0^\infty\frac{dv}{\sqrt{2\pi v^3}}\left(1-e^{-\rho v}\mathbb{I}_{v<D}\right)}\end{aligned}$$

where $q(r;u,v)$ is the density of a Bessel(3) bridge taken at time $u\leq v$ from 0 to 0 of length v.

A third proof of Chesney, Jeanblanc, and Yor's theorem

It is very quick to see that

$$\begin{aligned}\int_0^\infty dte^{-\rho t}\mathbb{E}\left[\mathbb{I}_{H_D^+>t}\right] &= \frac{1-\mathbb{E}\left[e^{-\rho H_D^+}\right]}{\rho} \\ &= \frac{\int_0^\infty \frac{dv}{\sqrt{2\pi v^3}}\int_0^v due^{-\rho u}\left(\mathbb{I}_{u<D}+1\right)}{\int_0^\infty \frac{dv}{\sqrt{2\pi v^3}}\left(1-e^{-\rho v}\right)+\int_0^\infty \frac{dv}{\sqrt{2\pi v^3}}\left(1-e^{-\rho v}\mathbb{I}_{v<D}\right)}\end{aligned}$$

so

$$\mathbb{E}\left[e^{-\rho H_D^+}\right] = 1-\rho\frac{\int_0^\infty \frac{dv}{\sqrt{2\pi v^3}}\int_0^v due^{-\rho u}\left(\mathbb{I}_{u<D}+1\right)}{\int_0^\infty \frac{dv}{\sqrt{2\pi v^3}}\left(1-e^{-\rho v}\right)+\int_0^\infty \frac{dv}{\sqrt{2\pi v^3}}\left(1-e^{-\rho v}\mathbb{I}_{v<D}\right)}.$$

Also, we have

$$\begin{aligned}&\int_0^\infty \frac{dv}{\sqrt{2\pi v^3}}\int_0^v due^{-\rho u}\left(\mathbb{I}_{u<D}+1\right) \\ &= \frac{1}{\rho}\int_0^D dv\frac{1-e^{-\rho v}}{\sqrt{2\pi v^3}}+\frac{1}{\rho}\int_0^\infty dv\frac{1-e^{-\rho v}}{\sqrt{2\pi v^3}}+\frac{1}{\rho}\left(1-e^{-\rho D}\right)\int_D^\infty \frac{dv}{\sqrt{2\pi v^3}}.\end{aligned}$$

In consequence, we write

$$\begin{aligned}\mathbb{E}\left[e^{-\rho H_D^+}\right] &= 1-\frac{\int_0^D dv\frac{1-e^{-\rho v}}{\sqrt{2\pi v^3}}+\int_0^\infty dv\frac{1-e^{-\rho v}}{\sqrt{2\pi v^3}}+\int_D^\infty dv\frac{1-e^{-\rho D}}{\sqrt{2\pi v^3}}}{\int_0^\infty \frac{dv}{\sqrt{2\pi v^3}}\left(1-e^{-\rho v}\right)+\int_0^D \frac{dv}{\sqrt{2\pi v^3}}\left(1-e^{-\rho v}\right)+\int_D^\infty \frac{dv}{\sqrt{2\pi v^3}}} \\ &= \frac{e^{-\rho D}\int_D^\infty \frac{dv}{\sqrt{2\pi v^3}}}{\int_0^\infty \frac{dv}{\sqrt{2\pi v^3}}\left(1-e^{-\rho v}\right)+\int_0^D \frac{dv}{\sqrt{2\pi v^3}}\left(1-e^{-\rho v}\right)+\int_D^\infty \frac{dv}{\sqrt{2\pi v^3}}}. \quad (3.2)\end{aligned}$$

With a change of variable and an integration by parts, we can show that

$$\begin{aligned}\frac{1}{\rho}\int_0^D dv\frac{1-e^{-\rho v}}{\sqrt{2\pi v^3}} &= \frac{2\sqrt{2}}{\sqrt{\rho}}\left(\frac{e^{-\rho D}-1}{\sqrt{2\pi}\sqrt{2\rho D}}+\int_0^{\sqrt{2\rho D}} dz\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\right) \text{ and} \\ \frac{1}{\rho}\int_0^\infty dv\frac{1-e^{-\rho v}}{\sqrt{2\pi v^3}} &= \frac{\sqrt{2}}{\sqrt{\rho}}.\end{aligned}$$

After replacing in 3.2, we have

$$\begin{aligned}\mathbb{E}\left[e^{-\rho H_D^+}\right] &= \frac{\frac{2e^{-\rho D}}{\sqrt{2\pi D}}}{\sqrt{2\rho}+2\sqrt{2\rho}\left(\frac{e^{-\rho D}-1}{\sqrt{2\pi}\sqrt{2\rho D}}+\mathcal{N}\left(\sqrt{2\rho D}\right)-\frac{1}{2}\right)+\frac{1}{\sqrt{2\pi D}}} \\ &= \frac{\frac{2e^{-\rho D}}{\sqrt{2\pi D}}}{2\frac{e^{-\rho D}}{\sqrt{2\pi D}D}+2\sqrt{2\rho}\mathcal{N}\left(\sqrt{2\rho D}\right)} \\ &= \frac{1}{1+\sqrt{2\pi D\rho}e^{-\rho D}\mathcal{N}\left(\sqrt{2\rho D}\right)} \\ &= \frac{1}{\phi\left(\sqrt{2\rho D}\right)}.\end{aligned}$$

And we therefore obtain a third proof of the theorem.

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Chapter 4 DECISION DELAYS IN ASYMETRIC DUAL COMPETITION[1]

By relying on the classical approach, an investment must be undertaken if and only if its expected discounted earnings are higher than its expected discounted costs. However, as we have shown in the preceding chapters, for now more than ten years, this approach neglects the risk and the irreversibility inherent in a lot of investment projects, and implicitly assumes that only two possibilities are available: investment now or never. The real option approach represents a much more realistic tool, since a third possibility is taken into account: wait and see... This approach fits very well the monopolist case. However, in a competitive case, the strategic behaviors of different firms are usually not taken into account. Some authors (see Lambrecht and Perraudin (1994, 1996)), have already tackled this problem and studied the preemption of investment projects. The aim of this chapter is also to consider this question, where unlike in the case treated by Lambrecht and Perraudin the competitors have different constraints in terms of investment delay and flexibility. In our setting, the large firm suffers a delay in its investment decisions, whereas the smaller firm's decisions are instantaneously implemented.

First of all, let us describe an example of a particular asymmetrical competition situation. We consider a competition situation, between a small firm and a big market leader. In the field of technology, between a specialized company and Microsoft for a new kind of user interface, or between a big pharmaceutical conglomerate and a biotechnologies research company to start the implementation of a new patent.

Under these conditions, it is natural to assume the small company knows the constraints of the big company, as the latter is highly "visible". But the contrary is not true, since the small company has probably no obligation to publish detailed material on its abilities and strategic plans. Moreover, investment decisions in a small structure can be considered to be instantaneous, whereas they would require a non negligible processing delay in a big multinational company. Finally, an important structure will benefit from scale economies, and the necessary sunk cost linked to an investment will be lower than for a smaller company. Also, the required return on investment by stockholders is lower for a large entity due to the diversification of its profits, that allows for more stable dividends.

We suppose, as it is the norm in such industries, that these investment decisions are not reversible, that is there is no option to disinvest. If the small company was contemplating to invest in a project, the future return of which is a stochastic variable, it is optimal for it to invest when this variable reaches a certain level. This corresponds to the usual real option theory. The large company, because of its decision making structure, would require a delay between the decision to invest and its implementation. In that case, we have shown in Chapter 2 that the best optimal decision process to implement is to invest when the future return spends continuously enough time over a given level to trigger the implementation. The

[1] A DIFFERENT VERSION OF THIS CHAPTER HAS ALSO BEEN CO-WRITTEN WITH MARC CHESNEY, USING A MORE ANALYTICAL APPROACH. THE TECHNICAL PART OF THIS CHAPTER THAT FOCUSES ON THE LENTH AND HEIGHT OF EXCURSIONS IS TO APPEAR IN THE *JOURNAL OF APPLIED PROBABILITY* IN DECEMBER 2002, UNDER THE TITLE "EXCURSION LENGTH AND HEIGHT AND APPLICATION TO FINANCE".

constant level in question, due to the lower costs of investing, will be substantially lower than the target level for the small company.

Therefore, we can expect the large company to find the optimum level that triggers its decision, knowing its constraints (costs, decision acceptance delay), but unaware of the small firm's intention. The small firm, then knowing when the large firm will invest, will find its own optimal investment level. From the small firm perspective, it is always optimal to preempt the larger one, as far as its break-even point is smaller than the investment threshold of the large firm. Indeed, if it does invest just before the large firm, it generates a positive value, versus a zero value if it does not invest.

We can also consider that the large firm is aware there is a potential competitor, but does not know its cost structure: only the distribution from which it is drown. Indeed, information about these potential small competitors would be accessible in an aggregate way, and the large firm would have rational expectations.

The paper is organized as follows: the first section is dedicated to presenting the analogy between Parisian American barrier options and the problem of valuing investment projects in an asymmetric competition situation as we have described above. This section follows an option pricing approach and reduces the problem to that of calculating the expectation of a functional of the Brownian Motion. In this section, the large firm is assumed to be unaware of the existence of a smaller competitor.

The second section focuses on a core result on various stopping times that help represent the preemption situation. This result is derived using excursion theory, following a method presented in Chapter 3. We assume here that the large firm follows the Parisian criterion because it is simple to implement, although it is not necessarily optimal. As we discussed in Chapter 2, an optimal strategy would involve the existence of a non-constant exercise frontier depending on the delay. An alternative approach has been developed in a working paper by Chesney and Gauthier (2001), that implicitly models this optimal exercise frontier for the large firm.

The third section studies the situation when the large firm is aware there is a competitor. Finally, the fourth section concludes this Chapter.

The model

Let us first consider the case when the small company acts strategically and tries to preempt the big one. The latter acts as if it was a monopolist and therefore doesn't act strategically. Both face only one investment opportunity, the same one. This is a situation of mutual preemption.

Following the lines of real option theory, we assume there exists an observed variable that conditions the future cashflows of the investment opportunity faced by the two competitors. We consider there are traded assets that can be used to perfectly replicate the value of this observed variable. Therefore we can use classical option theory to price the real options involved in the competition situation.

The dynamics of the observed variable, under the risk-neutral probability, solves the following SDE

$$\frac{dS_t}{S_t} = (r - \delta)\, dt + \sigma dW_t \tag{4.1}$$

where S_t, r, δ, σ, $(W_t, t \geqslant 0)$ are respectively the underlying value at time t, the domestic risk free rate, the convenience yield, the volatility, and a $\mathbb{Q}$-Brownian motion. We suppose the risk free rate and volatility are constant. In that frame, the results are equivalent if we suppose agents are risk-neutral, and the yield of the underlying value verifies $\mu = r - \delta$.

We will consider also that $S_0 = x$. So $S_t = x \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right)$, which we rewrite as $S_t = xe^{\sigma(mt+W_t)}$, or $S_t = xe^{\sigma Z_t^m}$ where Z^m is a drifted Brownian Motion, i.e. $(Z_t^m = mt + W_t, t \geq 0)$.

Let us define the following functionals

$$\begin{aligned} T_h(X) &= \inf\{t \geq 0 : X_t = h\} \\ g_t^h(X) &= \sup\{s \in [0,t] : X_s = h\} \\ H_{h,D}^+(X) &= \inf\left\{t \geq T_h : \left(t - g_t^h\right) \geq D \text{ and } X_t \geq h\right\}. \end{aligned}$$

They are, respectively, the first instant a process hits a given level, the last instant when the process was at a given level, and the Parisian time, the first instant when the process spends consecutively more than D units of time over a given level. Notice that $g_t^h(X)$ is not a stopping time. When this random time "happens", there is no way to know immediately that is has just happened..

Investment decision and options

We write the value of the project once launched, as

$$F(S_t, \infty) = \mathbb{E}_{\mathcal{F}_t} \int_t^\infty ds e^{-r(s-t)} f(S_s)$$

where f is a power function, thus allowing us to deal with Cobb-Douglas utility functions. Typically, in a simple case where f is the identity, we would have $F(S_0, \infty) = \Delta S_0$ with $\Delta = \frac{1}{\delta}$. Linked to the investment project, there are entry costs K_e and infinite exit costs. It means that once started the project cannot be stopped. The entry cost is therefore a sunk cost that cannot be recovered in the future. Also, we suppose there is no external reason why the firms should invest before any given time.

We can compute the level K corresponding to the break-even point. At that point we would have $K_e = \frac{K}{r}$, so $K = K_e r$.

The case of competition between companies with similar constraints has been studied by Dixit and Pindyck (1994) and Lambrecht and Perraudin (1994, 1996). The former show how in a competition situation, if there are many competitors the decision criterion is in fact the NPV. In the case of a duopoly, the latter have shown that the non availability of all information about one's opponent creates an equilibrium. In this case, the optimal investment barrier, even for a perpetual project, evolves through time depending on the rolling supremum reached by the observed variable: agents update their subjective probability distribution for the other's optimal barrier, and find in turn their own optimal level. This process allows for example a firm with a low cost to realize its opponent has a higher cost, and therefore a higher barrier, depending on how the observed variable will evolve.

If one firm is informed and the other does not even suspect its existence, then the optimal strategy for the informed is to invest at the first instant between the hitting time of its optimal level in monopoly, and the hitting time of its opponent's optimal level (or an infinitesimal time before), as far as it is above its own break-even point.

Competition between a large and a small companies

We will use the subscript 1 for the large company, and 2 for the small one. The small company will maximize its expected payoffs, its value at time zero, $V_2(x)$ is given by:

$$\begin{aligned} V_2(x) &= \sup_{L_2} \mathbb{E}\Big[\exp(-rT_{L_2})(L_2 - K_2)\,\mathbb{I}_{T_{L_2}<H^+_{L_1,D}} \\ &\quad + \exp\left(-rH^+_{L_1,D}\right)\left(S_{H^+_{L_1,D}} - K_2\right)\mathbb{I}_{T_{L_2}\geqslant H^+_{L_1,D}}\Big] \end{aligned} \tag{4.2}$$

where $x = S_0$, K_2 are the expected discounted costs generated by the investment project, for the small firm, L_2 is the underlying value level at which the firm will decide to invest, and where $H^+_{L_1,D}$ is the investment date of the big company, using the notations defined earlier.

Indeed, the large company invests only if the underlying value has remained during a period D above a given level L_1. Due to the delay in its decision process, the large company has interest in being able to cancel its investment decisions if market conditions deteriorate. The Parisian stopping time, though not the absolute optimal solution in that frame, is simple enough to be implemented easily and still allows a much better value for the firm than following the usual hitting-time linked investment decision.

If $T_{L_2} < H^+_{L_1,D}$ the small firm invests at its optimal level L_2, otherwise, it will also preempt the big company, even if it is not optimal from the monopolist point of vue. Indeed, it is better to generate positive earnings than to loose the project as far as the break-even point K_2 of the smaller firm is lower than the investment level for the large firm L_1. In this case, the investment will take place just before the big company optimal date $H^+_{L_1,D}$, at a level $S_{H^+_{L_1,D}}$ lower than L_2. In fact, the small firm will invest an instant arbitrarily small before the large one, so we can consider that it invests at the same time, but preempts the opportunity. We have represented both firms investment decisions in Figure 1, p. 57.

Let us now turn to the general problem of the valuation of American Parisian Options.

The up and out Parisian call

Definition 11 *An American up and out Parisian option gives its owner the right to exercise it at will until either maturity (which can be infinite) or until the first instant when the underlying price spends consecutively more than D units of time over a given level L_1 (the so-called Parisian time). If the investor decides to exercise at the stopping time τ, the present value of the payoff of the option writes*

$$e^{-r\tau}(S_\tau - K)^+\,\mathbb{I}_{\tau\leq H^+_{L_1,D}}\mathbb{I}_{\tau\leq T}$$

where T is the maturity and K the strike price. The value of this option is defined as $C^u_{Ao}(x,t)$ for an underlying x and a time to maturity t.

It is interesting to note that the value of the small firm $V_2(x)$ given by equation 4.2, corresponds exactly to the value of an up and out perpetual American Parisian Call Value (see Chesney, Jeanblanc, Yor (1997), for the European Parisian Options case) $C^u_{Ao}(S_t, T-t)$, where K_2, L_1 and D are respectively the strike price, the barrier and the window of the option. Indeed, the owner of such an option, will have to figure out the optimal exercise boundary (L_2 in our example) and to

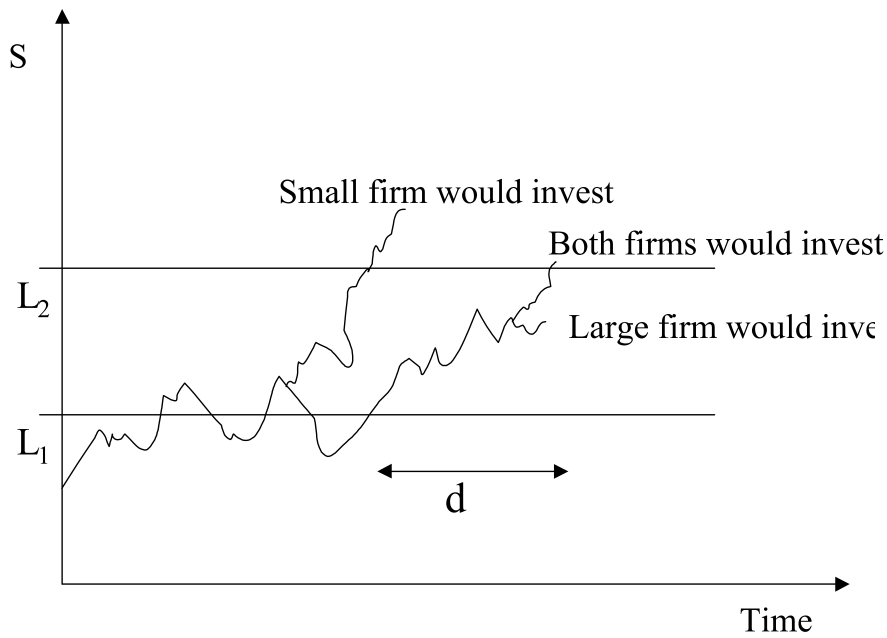

Figure 1 Large and Small Firms Investment Thresholds

exercise its option either at this level, if it is reached before the option is lost, or just before loosing its option.

In the perpetual case, the optimal strategy consists of exercising the option either at an optimal level (equivalent to the optimal level in the case of a standard perpetual American call) or just before it is cancelled, that is at the Parisian time if it happens before. Though, if the exercise price is above the Parisian threshold L_1, then it is not always optimal to exercise the option at the Parisian time. In that case, if the Parisian time happens before the optimal exercise barrier is hit, then depending on whether the underlying is above the strike price, the option is exercised or not.

Returning to the analogy with investment decision, if the small firm could not preempt the large one, and could only invest by following the hitting time rule, then the value of the investment project would correspond to another kind of option, that can be exercised only when the underlying hits an optimal level, fixed in the terms of the contract, and cancelled at the Parisian time. This option would have the following payoff (net present value)

$$e^{-r\tau}\left(S_\tau - K\right)^+ \mathbb{I}_{\tau \leq H^+_{L_1,D}} \mathbb{I}_{\tau \leq T}$$

with $\tau = T_L$, the hitting time of a given optimal level. We will call it the non preemptive American Parisian up and out call.

In this case, both the barrier and the exercise boundary are below the spot price. The option is therefore exercised either when the spot price reaches the exercise boundary L_1 , or just before the time at which the option is lost ($H^+_{L_1,D}$).

Proposition 12 *The value of the perpetual Parisian up and out call, with $K \leq L_1$ is equal to the supremum over L of*

$$V\left(S_0\right) \;=\; \mathbb{E}\left[\exp\left(-rT_L\right)\left(L-K\right)\mathbb{I}_{T_L<H^+_{L_1,D}} + \exp\left(-rH^+_{L_1,D}\right)\left(S_{H^+_{L_1,D}} - K\right)\mathbb{I}_{T_L \geqslant H^+_{L_1,D}}\right]$$

$$= \left(\frac{L_1}{S_0}\right)^{\sqrt{\frac{2r+m^2}{2\sigma^2}}} \frac{\left[(1-\mathsf{AB}) f(a) + \left(e^{a\sqrt{2\rho}}\mathsf{B} - 1\right)\frac{\mathsf{K}(f)}{\mathsf{K}(1)}\right]}{e^{a\sqrt{2\rho}} - \mathsf{A}}$$

with $m = \frac{1}{\sigma}\left(\mu - \frac{\sigma^2}{2}\right)$, $\rho = r + \frac{m^2}{2}$, $a = \frac{1}{\sigma}\ln\left(\frac{L}{L_1}\right)$, $f(z) = (L_1 e^{\sigma z} - K)\, e^{mz}$ *and* A, B, K *defined in Section 2.*

A decomposition appears between the part of the price that is due to a normal exercise

$$\left(\frac{L_1}{S_0}\right)^{\sqrt{\frac{2r+m^2}{2\sigma^2}}} \frac{(1-\mathsf{AB}) f(a)}{e^{a\sqrt{2\rho}} - \mathsf{A}}$$

and the part of the price linked to a forced early exercise at the Parisian time

$$\left(\frac{L_1}{S_0}\right)^{\sqrt{\frac{2r+m^2}{2\sigma^2}}} \frac{\left(e^{a\sqrt{2\rho}}\mathsf{B} - 1\right)\frac{\mathsf{K}(f)}{\mathsf{K}(1)}}{e^{a\sqrt{2\rho}} - \mathsf{A}}.$$

Proof. First of all, we want to write the price as a functional of the Brownian Motion. We have

$$\begin{aligned}
V(S_0) &= \mathbb{E}_{S_0}\left[\exp(-rT_L)(L-K)\,\mathbb{I}_{T_L < H^+_{L_1,D}}\right] \\
&\quad + \mathbb{E}_{S_0}\left[\exp\left(-rH^+_{L_1,D}\right)\left(S_{H^+_{L_1,D}} - K\right)\mathbb{I}_{T_L \geqslant H^+_{L_1,D}}\right] \\
&= \mathbb{E}_0\left[\exp(-rT_b(W^m))(L-K)\,\mathbb{I}_{T_b(W^m) < H^+_{b_1,D}(W^m)}\right] \\
&\quad + \mathbb{E}_0\left[\exp\left(-rH^+_{b_1,D}(W^m)\right)\left(S_0 e^{\sigma W^m_{H^+_{b_1,D}(W^m)}} - K\right)\mathbb{I}_{T_b(W^m) \geq H^+_{b_1,D}(W^m)}\right]
\end{aligned}$$

with

$$m = \frac{1}{\sigma}\left(r - \delta - \frac{\sigma^2}{2}\right)$$

and

$$\begin{aligned}
b &= \frac{1}{\sigma}\ln\left(\frac{L}{S_0}\right) \\
b_1 &= \frac{1}{\sigma}\ln\left(\frac{L_1}{S_0}\right)
\end{aligned}$$

and $W^m_t = mt + W_t$, a drifted Brownian Motion. In the expression above, we have written

$$S_{H^+_{L_1,D}} = S_0 e^{\sigma W^m_{H^+_{b_1,D}(W^m)}}$$

so as to be able to directly work with the Brownian Motion W^m. We can now apply Girsanov's theorem, and write

$$\begin{aligned}
V(S_0) &= \mathbb{E}_0\left[\exp(-rT_b(W))(L-K)\,\mathbb{I}_{T_b(W) < H^+_{b_1,D}(W)}\, e^{-\frac{m^2}{2}T_b(W) + mW_{T_b(W)}}\right] \\
&\quad + \mathbb{E}_0\left[\exp\left(-rH^+_{b_1,D}(W)\right)\left(S_0 e^{\sigma W_{H^+_{b_1,D}(W)}} - K\right)\right. \\
&\quad \left.\mathbb{I}_{T_b(W) \geq H^+_{b_1,D}(W)}\, e^{-\frac{m^2}{2}H^+_{b_1,D}(W) + mW_{H^+_{b_1,D}(W)}}\right]
\end{aligned}$$

$$\begin{aligned}
&= \mathbb{E}_0\left[\exp\left(-\left(r+\frac{m^2}{2}\right)T_b\right)\left(S_0 e^{\sigma W_{T_b}}-K\right)\mathbb{I}_{T_b<H^+_{b_1,D}}e^{mb}\right] \\
&\quad +\mathbb{E}_0\left[\exp\left(-\left(r+\frac{m^2}{2}\right)H^+_{b_1,D}\right)\left(S_0 e^{\sigma W_{H^+_{b_1,D}}}-K\right)\mathbb{I}_{T_b\geq H^+_{b_1,D}}e^{mW_{H^+_{b_1,D}}}\right] \\
&= \mathbb{E}_0\left[\exp\left(-\left(r+\frac{m^2}{2}\right)\tau\right)\left(S_0 e^{\sigma W_\tau}-K\right)e^{mW_\tau}\right]
\end{aligned}$$

where all the functionals refer to the (new) Brownian Motion W under the new probability, with $\tau = T_b \wedge H^+_{b_1,D}$. So as to simplify this expression, it is now natural to condition by the first hitting time of b_1, which will necessary be smaller than τ for $b \geq b_1$. Applying the strong Markov property at T_{b_1} gives therefore

$$\begin{aligned}
V(S_0) &= \mathbb{E}_0\left[\exp\left(-\left(r+\frac{m^2}{2}\right)\tau\right)\left(S_0 e^{\sigma W_\tau}-K\right)e^{mW_\tau}\right] \\
&= \mathbb{E}_0\left[\exp\left(-\left(r+\frac{m^2}{2}\right)T_{b_1}\right)\right] \\
&\quad \mathbb{E}_{b_1}\left[\exp\left(-\left(r+\frac{m^2}{2}\right)\tau\right)\left(S_0 e^{\sigma W_\tau}-K\right)e^{mW_\tau}\right] \\
&= \mathbb{E}_0\left[\exp\left(-\left(r+\frac{m^2}{2}\right)T_{b_1}\right)\right] \\
&\quad \mathbb{E}_0\left[\exp\left(-\left(r+\frac{m^2}{2}\right)T_{b-b_1}\wedge H^+_{0,D}\right)\left(L_1 e^{\sigma W_{T_{b-b_1}\wedge H^+_{0,D}}}-K\right)e^{mW_{T_{b-b_1}\wedge H^+_{0,D}}}\right].
\end{aligned}$$

We have the following well known Laplace transform:

$$\exp\left(-b_1\sqrt{2\left(r+\frac{m^2}{2}\right)}\right) = \left(\frac{L_1}{S_0}\right)^{\sqrt{\frac{2r+m^2}{2\sigma^2}}}$$

Now, thanks to theorem 13 on p. 60, we can easily write

$$\begin{aligned}
&\mathbb{E}_0\left[\exp\left(-\left(r+\frac{m^2}{2}\right)T_{b-b_1}\wedge H^+_{0,D}\right)\left(L_1 e^{\sigma W_{T_{b-b_1}\wedge H^+_{0,D}}}-K\right)e^{mW_{T_{b-b_1}\wedge H^+_{0,D}}}\right] \\
&= \mathbb{E}\left[e^{-\rho T_a\wedge H^+_D} f\left(W_{T_a\wedge H^+_D}\right)\right] = \frac{\left[(1-\mathsf{AB})\, f(a) + \left(e^{a\sqrt{2\rho}}\mathsf{B}-1\right)\frac{\mathsf{K}(f)}{\mathsf{K}(1)}\right]}{e^{a\sqrt{2\rho}}-\mathsf{A}}
\end{aligned}$$

with $\rho = r + \frac{m^2}{2}$, $a = b - b_1$, $f(z) = (L_1 e^{\sigma z} - K)\, e^{mz}$, and where A, B, K, depend on ρ. This ends the proof of the proposition. ■

The optimal value for the level L can be found by solving the first order condition equation

$$\frac{\partial V}{\partial L} = 0$$

for which a closed-form expression is difficult to obtain.

Note that if the firms' parameters are such that $\lim_{L\to\infty} V(L) = \infty$, then it means that the small firm will only invest just before the large firm, if the variable S is above its break-even threshold K_2 at the Parisian time.

The case of an already started excursion

In this case, we assume that, at time zero, the excursion has already started (so it started at a negative time). The impact of such a situation on the value of the option is in fact very simple: either the underlying price hits the barrier back again quickly enough in the remaining time, or it does not. If it does not, the option is cancelled at that time. So the option's value will be the value starting from the barrier level weighed by the probability that the underlying price hits it before it is too late.

More specifically, let us assume the excursion has started u units of time ago. Then, if the price does not go back up to L_1 before $D-u$ units of time, the option will be cancelled. At time zero, the underlying price S_0 has to be above L_1 since the excursion has started. So we can consider T_{L_1}, the hitting time of the barrier. When the price hits the barrier, we revert to the previous case again, and the price of the option is known: it is $C^u_{Ao}(L_1,+\infty)$. So, naturally, we obtain

$$C^u_{Ao}(S_0,+\infty,u)=\mathbb{E}_{S_0}\left[\mathbb{I}_{T_{L_1}\le u}e^{-rT_{L_1}}\right]C^u_{Ao}(L_1,+\infty).$$

We now turn to the proof of the formula we used to obtain a closed-form formula for the value of the small firm's investment project.

On the length and height of excursions

In this section, we are interested in the paths such that the Parisian time is triggered before or after the process hits a certain level. We will derive the Laplace transforms of stopping times involving the Parisian time and classical hitting times as well.

More precisely we are interested in the direct computation of the following quantities, for a Brownian Motion starting from 0:

$$\mathbb{E}\left[e^{-\rho T_a}\mathbb{I}_{H_D^+\ge T_a}\right] \tag{4.3}$$

$$\mathbb{E}\left[e^{-\rho H_D^+}\mathbb{I}_{H_D^+\le T_a}f\left(W_{H_D^+}\right)\right] \tag{4.4}$$

$$\mathbb{E}\left[e^{-\rho H_D^+\wedge T_a}\right] \tag{4.5}$$

where we consider these expectations for a Brownian Motion W starting from 0. To simplify notations we have written $H_D^+=H_{0,D}^+$. The first expression is used in the computation of the value of investing for the small company (and thus preempting the larger one). The second expression gives the value to the larger company, which is preempted. Finally, the third expression intervenes in the calculation of the two others. It will be calculated using excursion theory. But for a start, we give the following

Theorem 13 *We have the following relationships for B a standard Brownian Motion, ρ a positive number, a a positive level, and f a measurable function bounded below:*

$$\begin{aligned}
\mathbb{E}\left[e^{-\rho T_a}\mathbb{I}_{H_D^+\ge T_a}\right] &= \frac{1-\mathsf{AB}}{e^{a\sqrt{2\rho}}-\mathsf{A}}\\
\mathbb{E}\left[e^{-\rho H_D^+}\mathbb{I}_{H_D^+\le T_a}f\left(W_{H_D^+}\right)\right] &= \frac{\mathsf{K}(f)}{\mathsf{K}(1)}\frac{e^{a\sqrt{2\rho}}\mathsf{B}-1}{e^{a\sqrt{2\rho}}-\mathsf{A}}\\
\mathbb{E}\left[e^{-\rho T_a\wedge H_D^+}f\left(W_{T_a\wedge H_D^+}\right)\right] &= \frac{\left[(1-\mathsf{AB})f(a)+\left(e^{a\sqrt{2\rho}}\mathsf{B}-1\right)\frac{\mathsf{K}(f)}{\mathsf{K}(1)}\right]}{e^{a\sqrt{2\rho}}-\mathsf{A}}
\end{aligned}$$

with

$$
\begin{aligned}
\mathsf{K}(f) &= \sum_{k\in\mathbb{Z}} \int_0^{\frac{a}{\sqrt{D}}} dz f\left(z\sqrt{D}\right)\left(z+\frac{2ka}{\sqrt{D}}\right)\exp\left(-\frac{1}{2}\left(z+\frac{2ka}{\sqrt{D}}\right)^2\right), \\
\mathsf{A} &= \frac{\sum_{k\in\mathbb{Z}} \int_0^{\frac{a}{\sqrt{D}}} dz\left(z+\frac{2ka}{\sqrt{D}}\right)\exp\left(\sqrt{2\rho D}z-\frac{1}{2}\left(z+\frac{2ka}{\sqrt{D}}\right)^2\right)}{2\sum_{k\in\mathbb{Z}}\exp\left(-\frac{2k^2a^2}{D}\right)\left(1-\exp\left(-\frac{(4k+1)a^2}{2D}\right)\right)}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathsf{B} &= 1-\frac{\int_0^a \frac{dm}{m^2}\left(1-\mathbb{E}_0^{(3)}\left[e^{-\rho(T_m+T'_m)\wedge D}\right]\right)}{\int_0^a \frac{dm}{m^2}\left(1-\mathbb{E}_0^{(3)}\left[e^{-\rho(T_m+T'_m)}\mathbb{I}_{T_m+T'_m<D}\right]\right)+\frac{1}{a}+\sqrt{2\rho}} \\
&\quad -\frac{\int_0^\infty \frac{dv}{\sqrt{2\pi v^3}}\left(1-e^{-\rho v}\right)+\frac{1}{a}\left(1-\mathbb{E}_0^{(3)}\left[e^{-\rho T_a\wedge D}\right]\right)}{\int_0^a \frac{dm}{m^2}\left(1-\mathbb{E}_0^{(3)}\left[e^{-\rho(T_m+T'_m)}\mathbb{I}_{T_m+T'_m<D}\right]\right)+\frac{1}{a}+\sqrt{2\rho}}.
\end{aligned}
$$

where for independent Bessel-3 processes R and R' we have

$$
\begin{aligned}
&\mathbb{E}_0^{(3)}\left[e^{-\rho(T_m+T'_m)}\mathbb{I}_{T_m+T'_m<D}\right] \\
&= \frac{\pi^4}{4m^4}\int_0^D due^{-\rho u}\int_0^u dv\sum_{k\in\mathbb{Z}}(-1)^{k+1}k^2e^{-\frac{k^2\pi^2v}{2m^2}}\sum_{l\in\mathbb{Z}}(-1)^{l+1}l^2e^{-\frac{l^2\pi^2(u-v)}{2m^2}}, \\
&\mathbb{E}_0^{(3)}\left[e^{-\rho(T_m+T'_m)\wedge D}\right] \\
&= \frac{\pi^4}{4m^4}\int_0^D due^{-\rho u}\int_0^u dv\sum_{k\in\mathbb{Z}}(-1)^{k+1}k^2e^{-\frac{k^2\pi^2v}{2m^2}}\sum_{l\in\mathbb{Z}}(-1)^{l+1}l^2e^{-\frac{l^2\pi^2(u-v)}{2m^2}} \\
&\quad +e^{-\rho D}\frac{\pi^4}{4m^4}\int_D^\infty du\int_0^u dv\sum_{k\in\mathbb{Z}}(-1)^{k+1}k^2e^{-\frac{k^2\pi^2v}{2m^2}}\sum_{l\in\mathbb{Z}}(-1)^{l+1}l^2e^{-\frac{l^2\pi^2(u-v)}{2m^2}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}_0^{(3)}\left[e^{-\rho T_a\wedge D}\right] \\
&= \int_0^D dve^{-\rho v}\frac{\pi^2}{2a^2}\sum_{k\in\mathbb{Z}}(-1)^{k+1}k^2e^{-\frac{k^2\pi^2v}{2a^2}} \\
&\quad +e^{-\rho D}\int_D^\infty dv\frac{\pi^2}{2a^2}\sum_{k\in\mathbb{Z}}(-1)^{k+1}k^2e^{-\frac{k^2\pi^2v}{2a^2}}.
\end{aligned}
$$

Proof. The three expressions 4.3, 4.4, 4.5 are closely related as the third one will be helpful in computing the first one, which in turn will help calculate the second one. The proof of the theorem will be shown in the remaining of this section.

The third expression clearly comes from the two other ones.

Exploiting the links between the three expressions

Studying such a problem it is natural to try to use exponential martingales properties. First we write

$$
\mathbb{E}\left[e^{-\rho H_D^+\wedge T_a}\right]=\mathbb{E}\left[e^{-\rho T_a}\mathbb{I}_{H_D^+\geq T_a}\right]+\mathbb{E}\left[e^{-\rho H_D^+}\mathbb{I}_{H_D^+\leq T_a}\right]. \tag{4.6}
$$

This expression states the relationship between two of the expressions above.

Lemma 14 *For all positive a and α, we have*

$$\mathbb{E}\left[e^{-\frac{\alpha^2}{2}T_a}\mathbb{I}_{H_D^+\geq T_a}\right]=\frac{1-\mathsf{A}_\alpha\mathbb{E}\left[e^{-\frac{\alpha^2}{2}H_D^+\wedge T_a}\right]}{e^{\alpha a}-\mathsf{A}_\alpha},$$

with

$$\mathsf{A}_\alpha=\frac{\sum_{k\in\mathbb{Z}}\int_0^{\frac{a}{\sqrt{D}}}dz\left(z+\frac{2ka}{\sqrt{D}}\right)\exp\left(\alpha\sqrt{D}z-\frac{1}{2}\left(z+\frac{2ka}{\sqrt{D}}\right)^2\right)}{2\sum_{k\in\mathbb{Z}}\exp\left(-\frac{2k^2a^2}{D}\right)\left(1-\exp\left(-\frac{(4k+1)a^2}{2D}\right)\right)}.$$

Proof. Let us define the exponential martingale $\left(M_t=\exp\left(\alpha W_t-\frac{\alpha^2}{2}t\right),t\geq 0\right)$ for any α . If we apply Doob's martingale stopping theorem at $H_D^+\wedge T_a$ (with a positive so that $W_{H_D^+\wedge T_a}$ is positive), we obtain

$$\begin{aligned}1&=\mathbb{E}\left[e^{\alpha W_{H_D^+\wedge T_a}-\frac{\alpha^2}{2}H_D^+\wedge T_a}\right]\\&=\mathbb{E}\left[e^{\alpha W_{H_D^+}-\frac{\alpha^2}{2}H_D^+}\mathbb{I}_{H_D^+\leq T_a}\right]+e^{\alpha a}\mathbb{E}\left[e^{-\frac{\alpha^2}{2}T_a}\mathbb{I}_{H_D^+\geq T_a}\right].\end{aligned}$$

This gives a relationship between 4.3 and 4.4, for a particular case of function f.

We concentrate now on the term $\mathbb{E}\left[e^{\alpha W_{H_D^+}-\frac{\alpha^2}{2}H_D^+}\mathbb{I}_{H_D^+\leq T_a}\right]$. We know that H_D^+ and (W_t) for $g\left(H_D^+\right)\leq t\leq H_D^+$ are independent, and as a consequence, $\sup_{g\left(H_D^+\right)\leq t\leq H_D^+}W_t$ and $W_{H_D^+}$ are independent of H_D^+. Also, we can write

$$\begin{aligned}&\mathbb{E}\left[e^{\alpha W_{H_D^+}-\frac{\alpha^2}{2}H_D^+}\mathbb{I}_{H_D^+\leq T_a}\right]\\&=\mathbb{E}\left[e^{\alpha W_{H_D^+}-\frac{\alpha^2}{2}H_D^+}\mathbb{I}_{\sup_{0\leq t\leq g\left(H_D^+\right)}W_t\leq a}\mathbb{I}_{\sup_{g\left(H_D^+\right)\leq t\leq H_D^+}W_t\leq a}\right]\\&=\mathbb{E}\left[\left(e^{-\frac{\alpha^2}{2}H_D^+}\mathbb{I}_{\sup_{0\leq t\leq g\left(H_D^+\right)}W_t\leq a}\right)\left(e^{\alpha W_{H_D^+}}\mathbb{I}_{\sup_{g\left(H_D^+\right)\leq t\leq H_D^+}W_t\leq a}\right)\right]\end{aligned}$$

and using the independence result,

$$\begin{aligned}&\mathbb{E}\left[e^{\alpha W_{H_D^+}-\frac{\alpha^2}{2}H_D^+}\mathbb{I}_{H_D^+\leq T_a}\right]\\&=\mathbb{E}\left[e^{-\frac{\alpha^2}{2}H_D^+}\mathbb{I}_{\sup_{0\leq t\leq g\left(H_D^+\right)}W_t\leq a}\right]\mathbb{E}\left[e^{\alpha W_{H_D^+}}\mathbb{I}_{\sup_{g\left(H_D^+\right)\leq t\leq H_D^+}W_t\leq a}\right].\end{aligned}$$

Also, we know that the trajectory $\left(W_t,g\left(H_D^+\right)\leq t\leq H_D^+\right)$ is a Brownian meander, and by scaling

$$\left(W_t,g\left(H_D^+\right)\leq t\leq H_D^+\right)=\left(\sqrt{D}m_u,0\leq u\leq 1\right)\text{ in law.}$$

Now, using Theorem 31 in Chapter 7 (p. 131) on the joint law of the Brownian Meander and its maximum, we write

$$\mathbb{E}\left[e^{\alpha W_{H_D^+}}\mathbb{I}_{\sup_{g\left(H_D^+\right)\leq t\leq H_D^+}W_t\leq a}\right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[e^{\alpha\sqrt{D}m_1}\mathbb{I}_{\sup_{0\le u\le 1} m_u \le \frac{a}{\sqrt{D}}}\right] \\
&= \sum_{k\in\mathbb{Z}} \int_0^{\frac{a}{\sqrt{D}}} dz \left(z+\frac{2ka}{\sqrt{D}}\right)\exp\left(\alpha\sqrt{D}z - \frac{1}{2}\left(z+\frac{2ka}{\sqrt{D}}\right)^2\right).
\end{aligned}
$$

Remark 2 *This calculation is also valid for all positive f. We have*

$$
\begin{aligned}
& \mathbb{E}\left[f\left(W_{H_D^+}\right)\mathbb{I}_{\sup_{g\left(H_D^+\right)\le t\le H_D^+} W_t\le a}\right] \\
&= \mathbb{E}\left[f\left(m_1\sqrt{D}\right)\mathbb{I}_{\sup_{0\le u\le 1} m_u\le \frac{a}{\sqrt{D}}}\right] \\
&= \sum_{k\in\mathbb{Z}} \int_0^{\frac{a}{\sqrt{D}}} dz f\left(z\sqrt{D}\right)\left(z+\frac{2ka}{\sqrt{D}}\right)\exp\left(-\frac{1}{2}\left(z+\frac{2ka}{\sqrt{D}}\right)^2\right) \overset{def}{=} \mathsf{K}(f).
\end{aligned}
$$

To explicit the term $\mathbb{E}\left[e^{-\frac{\alpha^2}{2}H_D^+}\mathbb{I}_{\sup_{0\le t\le g\left(H_D^+\right)} W_t\le a}\right]$ we write

$$
\begin{aligned}
& \mathbb{E}\left[e^{-\frac{\alpha^2}{2}H_D^+}\mathbb{I}_{H_D^+\le T_a}\right] \\
&= \mathbb{E}\left[e^{-\frac{\alpha^2}{2}H_D^+}\mathbb{I}_{\sup_{0\le t\le g\left(H_D^+\right)} W_t\le a}\mathbb{I}_{\sup_{g\left(H_D^+\right)\le t\le H_D^+} W_t\le a}\right] \\
&= \mathbb{E}\left[e^{-\frac{\alpha^2}{2}H_D^+}\mathbb{I}_{\sup_{0\le t\le g\left(H_D^+\right)} W_t\le a}\right]\mathbb{E}\left[\mathbb{I}_{\sup_{g\left(H_D^+\right)\le t\le H_D^+} W_t\le a}\right]
\end{aligned}
$$

and therefore, using the theorem again, we get

$$
\begin{aligned}
& \mathbb{E}\left[e^{-\frac{\alpha^2}{2}H_D^+}\mathbb{I}_{\sup_{0\le t\le g\left(H_D^+\right)} W_t\le a}\right] \\
&= \frac{\mathbb{E}\left[e^{-\frac{\alpha^2}{2}H_D^+}\mathbb{I}_{H_D^+\le T_a}\right]}{\sum_{k\in\mathbb{Z}}\int_0^{\frac{a}{\sqrt{D}}} dz\left(z+\frac{2ka}{\sqrt{D}}\right)\exp\left(-\frac{1}{2}\left(z+\frac{2ka}{\sqrt{D}}\right)^2\right)} \\
&= \frac{\mathbb{E}\left[e^{-\frac{\alpha^2}{2}H_D^+}\mathbb{I}_{H_D^+\le T_a}\right]}{2\sum_{k\in\mathbb{Z}}\exp\left(-\frac{2k^2a^2}{D}\right)\left(1-\exp\left(-\frac{(4k+1)a^2}{2D}\right)\right)}.
\end{aligned}
$$

Gathering the results, we have

$$
\begin{aligned}
1 &= \frac{\sum_{k\in\mathbb{Z}}\int_0^{\frac{a}{\sqrt{D}}} dz\left(z+\frac{2ka}{\sqrt{D}}\right)\exp\left(\alpha\sqrt{D}z-\frac{1}{2}\left(z+\frac{2ka}{\sqrt{D}}\right)^2\right)}{2\sum_{k\in\mathbb{Z}}\exp\left(-\frac{2k^2a^2}{D}\right)\left(1-\exp\left(-\frac{(4k+1)a^2}{2D}\right)\right)} \\
&\quad \mathbb{E}\left[e^{-\frac{\alpha^2}{2}H_D^+}\mathbb{I}_{H_D^+\le T_a}\right] + e^{\alpha a}\mathbb{E}\left[e^{-\frac{\alpha^2}{2}T_a}\ \mathbb{I}_{H_D^+\ge T_a}\right].
\end{aligned} \tag{4.7}
$$

This gives a second relationship between the quantities we are interested in. For notational convenience we will write 4.7 as

$$
1 = \mathsf{A}_\alpha\mathbb{E}\left[e^{-\frac{\alpha^2}{2}H_D^+}\mathbb{I}_{H_D^+\le T_a}\right] + e^{\alpha a}\mathbb{E}\left[e^{-\frac{\alpha^2}{2}T_a}\mathbb{I}_{H_D^+\ge T_a}\right]
$$

so when combined with 4.6 we get

$$\mathbb{E}\left[e^{-\frac{\alpha^2}{2}T_a}\mathbb{I}_{H_D^+\geq T_a}\right] = \frac{1-\mathsf{A}_\alpha\mathbb{E}\left[e^{-\frac{\alpha^2}{2}H_D^+\wedge T_a}\right]}{e^{\alpha a}-\mathsf{A}_\alpha},$$

as stated in the Lemma. ■

Reformulation of $\mathbb{E}\left[e^{-\rho H_D^+\wedge T_a}\right]$

Lemma 15 *For all positive ρ, we have*

$$\mathbb{E}\left[e^{-\rho H_D^+\wedge T_a}\right] = 1-\rho\frac{\int \mathbf{n}\left(d\varepsilon\right)\int_0^{V(\varepsilon)} dve^{-\rho v}\left(\mathbb{I}_{\varepsilon_v>0}\mathbb{I}_{v<D}+\mathbb{I}_{\varepsilon_v\leq 0}\right)\left(\mathbb{I}_{\varepsilon_v>0}\mathbb{I}_{\sup_{u\leq v}\varepsilon_u\leq a}+\mathbb{I}_{\varepsilon_v\leq 0}\right)}{\int \mathbf{n}\left(d\varepsilon\right)\left(1-e^{-\rho V(\varepsilon)}\left(\mathbb{I}_{\varepsilon\leq 0}+\mathbb{I}_{\varepsilon>0}\mathbb{I}_{V(\varepsilon)<D}\right)\left(\quad\mathbb{I}_{\varepsilon\leq 0}+\mathbb{I}_{\varepsilon>0}\mathbb{I}_{\sup_{0\leq u\leq V(\varepsilon)}\varepsilon_u<a}\right)\right)}$$

Proof. To compute this quantity we follow the approach lined out in Chapter 3, Section 4. First of all, we define $\tau = H_D^+\wedge T_a$, and write

$$\mathbb{E}\left[\int_0^\infty dte^{-\rho t}\mathbb{I}_{\tau\geq t}\right] = \frac{1}{\rho}\left(1-\mathbb{E}\left[e^{-\rho\tau}\right]\right)$$

so

$$\mathbb{E}\left[e^{-\rho\tau}\right] = 1-\rho\mathbb{E}\left[\int_0^\infty dte^{-\rho t}\mathbb{I}_{\tau\geq t}\right].$$

Now, as before, let us also define the longest positive excursion up to a time τ_{s-} as

$$l^+\left(\tau_{s-}\right) = \sup\left\{l\geq 0 : \exists u, u<s, \left(\tau_u-\tau_{u-}\right)=l, \mathbf{e}_u\geq 0\right\}$$

and we introduce the highest level reached by an excursion up to time τ_{s-} defined by

$$h^+\left(\tau_{s-}\right) = \sup\left\{h\geq 0 : \exists u, u<s, \sup_{\tau_{u-}\leq v\leq\tau_u}\mathbf{e}_u\left(v\right)=l, \mathbf{e}_u\geq 0\right\}$$

where $\mathbf{e}$ is the excursion process. These random variables can also be defined up to the last zero as

$$l^+\left(g_t\right) = \sup\left\{\left(d_s-g_s\right) : g_t>s\geq 0, W_s\geq 0\right\}$$

and

$$h^+\left(g_t\right) = \sup\left\{W_s : g_t>s\geq 0, W_s\geq 0\right\}.$$

V will be used to denote the length of an excursion, that is $V\left(\mathbf{e}_s\right)=\tau_s-\tau_{s-}$.

Now, let us notice we have the following equalities of events

$$\begin{aligned}
(\tau>t) &= \left(D>l^+\left(g_t\right)\right) \\
&\bigcap\left(\left(W_t\leq 0\right)\bigcup\left(\left(W_t>0\right)\bigcap\left(t-g_t<D\right)\right)\right) \\
&\bigcap\left(h^+\left(g_t\right)<a\right) \\
&\bigcap\left(\left(W_t\leq 0\right)\bigcup\left(\left(W_t>0\right)\bigcap\sup_{g_t\leq u\leq t}W_u\leq a\right)\right).
\end{aligned}$$

We deduce that we can write

$$\begin{aligned}
&\int_0^\infty dt\mathbb{E}\left[e^{-\rho t}\mathbb{I}_{\tau\leq t}\right]\\
= \ & \mathbb{E}\int_0^\infty dte^{-\rho t}\mathbb{I}_{l^+(g_t)<D}\left(\mathbb{I}_{W_t>0}\mathbb{I}_{t-g_t<D}+\mathbb{I}_{W_t\leq 0}\right)\\
&\times\mathbb{I}_{h^+(g_t)<a}\left(\mathbb{I}_{W_t>0}\mathbb{I}_{\sup_{g_t\leq u\leq t}W_u\leq a}+\mathbb{I}_{W_t\leq 0}\right).
\end{aligned}$$

Let us recall the balayage theory result mentioned in Chapter 3:

$$\begin{aligned}
&\mathbb{E}\left[\int_0^\infty dte^{-\rho t}F_1\left(B_u,u\leq g_t\right)F_2\left(B_u,g_t\leq u\leq t\right)\right]\\
= \ & \mathbb{E}\left[\int_0^\infty dse^{-\rho\tau_{s-}}F_1\left(B_u,u\leq\tau_{s-}\right)\right]\int\mathbf{n}\left(d\varepsilon\right)\int_0^{V(\varepsilon)}dve^{-\rho v}F_2\left(\varepsilon_u,u\leq v\right).
\end{aligned}$$

Applying it to the particular case we are studying entails immediately

$$\begin{aligned}
&\mathbb{E}\int_0^\infty dte^{-\rho t}\mathbb{I}_{l^+(g_t)<D}\left(\mathbb{I}_{W_t>0}\mathbb{I}_{t-g_t<D}+\mathbb{I}_{W_t\leq 0}\right)\\
&\mathbb{I}_{h^+(g_t)<a}\left(\mathbb{I}_{W_t>0}\mathbb{I}_{\sup_{g_t\leq u\leq t}W_u\leq a}+\mathbb{I}_{W_t\leq 0}\right)\\
= \ & \mathbb{E}\left[\int_0^\infty dse^{-\rho\tau_{s-}}\mathbb{I}_{l^+(\tau_{s-})<D}\mathbb{I}_{h^+(\tau_{s-})<a}\right]\\
&\int\mathbf{n}\left(d\varepsilon\right)\int_0^{V(\varepsilon)}dve^{-\rho v}\left(\mathbb{I}_{\varepsilon_v>0}\mathbb{I}_{v<D}+\mathbb{I}_{\varepsilon_v\leq 0}\right)\left(\mathbb{I}_{\varepsilon_v>0}\mathbb{I}_{\sup_{u\leq v}\varepsilon_u\leq a}+\mathbb{I}_{\varepsilon_v\leq 0}\right).
\end{aligned}$$

We will use William's description of $\mathbf{n}$, based on the maximum of the excursion.

- $\mathbf{n}\left(\sup_{u\leq V(\varepsilon)}\varepsilon_u\in dm\right)=\frac{dm}{m^2}$
- The excursion path conditionally to m is composed of two Bessel-3 processes put back to back between 0 and their hitting times of m.

Let us also recall Itô's description of the measure $\mathbf{n}$: $\mathbf{n}\left(V\left(\varepsilon\right)\in dv\right)=\frac{dv}{\sqrt{2\pi v^3}}$. We now turn to the computation of the path integral. We are interested in

$$\mathbb{E}\left[\int_0^\infty dse^{-\rho\tau_{s-}}\mathbb{I}_{l^+(\tau_{s-})<D}\mathbb{I}_{h^+(\tau_{s-})<a}\right].$$

We start with

$$e^{-\rho\tau_{s-}}\mathbb{I}_{l^+(\tau_{s-})<D}\mathbb{I}_{h^+(\tau_{s-})<a}=1+\sum_{0\leq u<s}\left(e^{-\rho\tau_u}\mathbb{I}_{l^+(\tau_u)<D}\mathbb{I}_{h^+(\tau_u)<a}-e^{-\rho\tau_{u-}}\mathbb{I}_{l^+(\tau_{u-})<D}\mathbb{I}_{h^+(\tau_{u-})<a}\right).$$

But we also write easily that

$$\begin{aligned}
&\left(l^+\left(\tau_u\right)<D\right)\bigcap\left(h^+\left(\tau_u\right)<a\right)\\
= \ & \left(l^+\left(\tau_{u-}\right)<D\right)\bigcap\left(\left(\mathbf{e}_u\leq 0\right)\bigcup\left(\left(\mathbf{e}_u>0\right)\bigcap\left(\tau_u-\tau_{u-}<D\right)\right)\right)\\
&\bigcap\left(h^+\left(\tau_{u-}\right)<a\right)\bigcap\left(\left(\mathbf{e}_u\leq 0\right)\bigcup\left(\left(\mathbf{e}_u>0\right)\bigcap\sup_{\tau_{u-}\leq v\leq\tau_u}\mathbf{e}_u\left(v\right)<a\right)\right)
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
& e^{-\rho\tau_{s-}}\mathbb{I}_{l^{+}(\tau_{s-})<D}\mathbb{I}_{h^{+}(\tau_{s-})<a} \\
= \;& 1+\sum_{0\leq u<s}\mathbb{I}_{l^{+}(\tau_{u})<D}\mathbb{I}_{h^{+}(\tau_{u})<a} \\
& \left(e^{-\rho\tau_{u}}\left(\mathbb{I}_{\mathbf{e}_{u}\leq 0}+\mathbb{I}_{\mathbf{e}_{u}>0}\mathbb{I}_{\tau_{u}-\tau_{u-}<D}\right)\left(\mathbb{I}_{\mathbf{e}_{u}\leq 0}+\mathbb{I}_{\mathbf{e}_{u}>0}\mathbb{I}_{\sup_{\tau_{u-}\leq v\leq\tau_{u}}\mathbf{e}_{u}(v)<a}\right)-e^{-\rho\tau_{u-}}\right) \\
= \;& 1+\sum_{0\leq u<s}e^{-\rho\tau_{u-}}\mathbb{I}_{l^{+}(\tau_{u-})<D}\mathbb{I}_{h^{+}(\tau_{u-})<a} \\
& \left(e^{-\rho V(\mathbf{e}_{s})}\left(\mathbb{I}_{\mathbf{e}_{u}\leq 0}+\mathbb{I}_{\mathbf{e}_{u}>0}\mathbb{I}_{\tau_{u}-\tau_{u-}<D}\right)\left(\mathbb{I}_{\mathbf{e}_{u}\leq 0}+\mathbb{I}_{\mathbf{e}_{u}>0}\mathbb{I}_{\sup_{\tau_{u-}\leq v\leq\tau_{u}}\mathbf{e}_{u}(v)<a}\right)-1\right).
\end{aligned}$$

Taking the expectation and applying the Compensation Formula yields

$$\begin{aligned}
& \mathbb{E}\left[e^{-\rho\tau_{s-}}\mathbb{I}_{l^{+}(\tau_{s-})<D}\mathbb{I}_{h^{+}(\tau_{s-})<a}\right] \\
= \;& 1+\mathbb{E}\left[\int_{0}^{s}due^{-\rho\tau_{u-}}\mathbb{I}_{l^{+}(\tau_{u-})<D}\mathbb{I}_{h^{+}(\tau_{u-})<a}\right] \\
& \int\mathbf{n}\left(d\varepsilon\right)\left(e^{-\rho V(\varepsilon)}\left(\mathbb{I}_{\varepsilon\leq 0}+\mathbb{I}_{\varepsilon>0}\mathbb{I}_{V(\varepsilon)<D}\right)\left(\mathbb{I}_{\varepsilon\leq 0}+\mathbb{I}_{\varepsilon>0}\mathbb{I}_{\sup_{0\leq u\leq V(\varepsilon)}\varepsilon_{u}<a}\right)-1\right).
\end{aligned}$$

If we define

$$\varphi_{D}\left(s\right)=\mathbb{E}\left[e^{-\rho\tau_{s-}}\mathbb{I}_{l^{+}(\tau_{s-})<D}\mathbb{I}_{h^{+}(\tau_{s-})<a}\right]$$

then we know that

$$\mathbb{E}\left[\int_{0}^{\infty}dse^{-\rho\tau_{s-}}\mathbb{I}_{l^{+}(\tau_{s-})<D}\mathbb{I}_{h^{+}(\tau_{s-})<a}\right]=\int_{0}^{\infty}ds\varphi_{D}\left(s\right)$$

and

$$\begin{aligned}
\varphi_{D}\left(s\right) = \;& 1+\int\mathbf{n}\left(d\varepsilon\right) \\
& \left(e^{-\rho V(\varepsilon)}\left(\mathbb{I}_{\varepsilon\leq 0}+\mathbb{I}_{\varepsilon>0}\mathbb{I}_{V(\varepsilon)<D}\right)\left(\mathbb{I}_{\varepsilon\leq 0}+\mathbb{I}_{\varepsilon>0}\mathbb{I}_{\sup_{0\leq u\leq V(\varepsilon)}\varepsilon_{u}<a}\right)-1\right)\int_{0}^{s}du\varphi_{D}\left(u\right).
\end{aligned}$$

Solving the differential equation gives directly

$$\begin{aligned}
& \mathbb{E}\left[\int_{0}^{\infty}dse^{-\rho\tau_{s-}}\mathbb{I}_{l^{+}(\tau_{s-})<D}\mathbb{I}_{h^{+}(\tau_{s-})<a}\right] \\
= \;& \frac{1}{\int\mathbf{n}\left(d\varepsilon\right)\left(1-e^{-\rho V(\varepsilon)}\left(\mathbb{I}_{\varepsilon\leq 0}+\mathbb{I}_{\varepsilon>0}\mathbb{I}_{V(\varepsilon)<D}\right)\left(\mathbb{I}_{\varepsilon\leq 0}+\mathbb{I}_{\varepsilon>0}\mathbb{I}_{\sup_{0\leq u\leq V(\varepsilon)}\varepsilon_{u}<a}\right)\right)}.
\end{aligned}$$

Rewriting the entire expression gives

$$\mathbb{E}\left[e^{-\rho\tau}\right]=1-\rho\frac{\int\mathbf{n}\left(d\varepsilon\right)\int_{0}^{V(\varepsilon)}dve^{-\rho v}\left(\mathbb{I}_{\varepsilon_{v}>0}\mathbb{I}_{v<D}+\mathbb{I}_{\varepsilon_{v}\leq 0}\right)\left(\mathbb{I}_{\varepsilon_{v}>0}\mathbb{I}_{\sup_{u\leq v}\varepsilon_{u}\leq a}+\mathbb{I}_{\varepsilon_{v}\leq 0}\right)}{\int\mathbf{n}\left(d\varepsilon\right)\left(1-e^{-\rho V(\varepsilon)}\left(\mathbb{I}_{\varepsilon\leq 0}+\mathbb{I}_{\varepsilon>0}\mathbb{I}_{V(\varepsilon)<D}\right)\left(\mathbb{I}_{\varepsilon\leq 0}+\mathbb{I}_{\varepsilon>0}\mathbb{I}_{\sup_{0\leq u\leq V(\varepsilon)}\varepsilon_{u}<a}\right)\right)}.$$

This result ends the proof. ■

We now turn to the actual calculation of these integrals.

Explicit computation of Itô measure integrals
We summarize the result in this

Lemma 16 *For all positive ρ*

$$\begin{aligned}
&\mathbb{E}\left[e^{-\rho\tau}\right] \\
&= 1-\frac{\int_0^a \frac{dm}{m^2}\left(1-\mathbb{E}_0^{(3)}\left[e^{-\rho(T_m+T'_m)\wedge D}\right]\right)}{\int_0^a \frac{dm}{m^2}\left(1-\mathbb{E}_0^{(3)}\left[e^{-\rho(T_m+T'_m)}\mathbb{I}_{T_m+T'_m<D}\right]\right)+\frac{1}{a}+\sqrt{2\rho}} \\
&\quad -\frac{\int_0^\infty \frac{dv}{\sqrt{2\pi v^3}}\left(1-e^{-\rho v}\right)+\frac{1}{a}\left(1-\mathbb{E}_0^{(3)}\left[e^{-\rho T_a\wedge D}\right]\right)}{\int_0^a \frac{dm}{m^2}\left(1-\mathbb{E}_0^{(3)}\left[e^{-\rho(T_m+T'_m)}\mathbb{I}_{T_m+T'_m<D}\right]\right)+\frac{1}{a}+\sqrt{2\rho}}.
\end{aligned}$$

with

$$\begin{aligned}
&\mathbb{E}_0^{(3)}\left[e^{-\rho(T_m+T'_m)}\mathbb{I}_{T_m+T'_m<D}\right] \\
&= \frac{\pi^4}{4m^4}\int_0^D due^{-\rho u}\int_0^u dv\sum_{k\in\mathbb{Z}}(-1)^{k+1}k^2e^{-\frac{k^2\pi^2 v}{2m^2}}\sum_{l\in\mathbb{Z}}(-1)^{l+1}l^2e^{-\frac{l^2\pi^2(u-v)}{2m^2}}, \\
&\mathbb{E}_0^{(3)}\left[e^{-\rho(T_m+T'_m)\wedge D}\right] \\
&= \frac{\pi^4}{4m^4}\int_0^D due^{-\rho u}\int_0^u dv\sum_{k\in\mathbb{Z}}(-1)^{k+1}k^2e^{-\frac{k^2\pi^2 v}{2m^2}}\sum_{l\in\mathbb{Z}}(-1)^{l+1}l^2e^{-\frac{l^2\pi^2(u-v)}{2m^2}} \\
&\quad +e^{-\rho D}\frac{\pi^4}{4m^4}\int_D^\infty du\int_0^u dv\sum_{k\in\mathbb{Z}}(-1)^{k+1}k^2e^{-\frac{k^2\pi^2 v}{2m^2}}\sum_{l\in\mathbb{Z}}(-1)^{l+1}l^2e^{-\frac{l^2\pi^2(u-v)}{2m^2}}
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}_0^{(3)}\left[e^{-\rho T_a\wedge D}\right] \\
&= \int_0^D dve^{-\rho v}\frac{\pi^2}{2a^2}\sum_{k\in\mathbb{Z}}(-1)^{k+1}k^2e^{-\frac{k^2\pi^2 v}{2a^2}} \\
&\quad +e^{-\rho D}\int_D^\infty dv\frac{\pi^2}{2a^2}\sum_{k\in\mathbb{Z}}(-1)^{k+1}k^2e^{-\frac{k^2\pi^2 v}{2a^2}}.
\end{aligned}$$

Proof. So as to compute these integrals, we have to study the "joint law" under Itô's measure of the supremum and the length of an excursion. Also, we need the joint law of a hitting time and the length of the excursion. First, let us see how these laws intervene in our problem. As for the denominator:

$$\begin{aligned}
&\int \mathbf{n}(d\varepsilon)\left(1-e^{-\rho V(\varepsilon)}\left(\mathbb{I}_{\varepsilon\le 0}+\mathbb{I}_{\varepsilon>0}\mathbb{I}_{V(\varepsilon)<D}\right)\left(\mathbb{I}_{\varepsilon\le 0}+\mathbb{I}_{\varepsilon>0}\mathbb{I}_{\sup_{0\le u\le V(\varepsilon)}\varepsilon_u<a}\right)\right) \\
&= \int \mathbf{n}(d\varepsilon)\left(1-e^{-\rho V(\varepsilon)}\mathbb{I}_{\varepsilon\le 0}-e^{-\rho V(\varepsilon)}\mathbb{I}_{\varepsilon>0}\mathbb{I}_{V(\varepsilon)<D}\mathbb{I}_{\sup_{0\le u\le V(\varepsilon)}\varepsilon_u<a}\right) \\
&= \int \mathbf{n}(d\varepsilon)\mathbb{I}_{\varepsilon\le 0}\left(1-e^{-\rho V(\varepsilon)}\right)+\int \mathbf{n}(d\varepsilon)\mathbb{I}_{\varepsilon>0}\left(1-e^{-\rho V(\varepsilon)}\mathbb{I}_{V(\varepsilon)<D}\mathbb{I}_{\sup_{0\le u\le V(\varepsilon)}\varepsilon_u<a}\right)
\end{aligned}$$

and for the numerator:

$$\begin{aligned}
&\int \mathbf{n}(d\varepsilon)\int_0^{V(\varepsilon)} dve^{-\rho v}\left(\mathbb{I}_{\varepsilon_v>0}\mathbb{I}_{v<D}+\mathbb{I}_{\varepsilon_v\le 0}\right)\left(\mathbb{I}_{\varepsilon_v>0}\mathbb{I}_{\sup_{u\le v}\varepsilon_u\le a}+\mathbb{I}_{\varepsilon_v\le 0}\right) \\
&= \int \mathbf{n}(d\varepsilon)\int_0^{V(\varepsilon)} dve^{-\rho v}\left(\mathbb{I}_{\varepsilon_v>0}\mathbb{I}_{v<D}\mathbb{I}_{\sup_{u\le v}\varepsilon_u\le a}+\mathbb{I}_{\varepsilon_v\le 0}\right).
\end{aligned}$$

In these integrals, the difficult terms are respectively

$$\int \mathbf{n}\,(d\varepsilon)\,\mathbb{I}_{\varepsilon>0}\left(1-e^{-\rho V(\varepsilon)}\mathbb{I}_{V(\varepsilon)<D}\mathbb{I}_{\sup_{0\leq u\leq V(\varepsilon)}\varepsilon_u<a}\right) \tag{4.8}$$

and

$$\begin{aligned}
&\int \mathbf{n}\,(d\varepsilon)\int_0^{V(\varepsilon)} dv e^{-\rho v}\mathbb{I}_{\varepsilon_v>0}\mathbb{I}_{v<D}\mathbb{I}_{\sup_{u\leq v}\varepsilon_u\leq a} \\
= &\int \mathbf{n}\,(d\varepsilon)\int_0^{V(\varepsilon)\wedge D\wedge T_a(\varepsilon)} dv e^{-\rho v}\mathbb{I}_{\varepsilon_v>0} \\
= &\frac{1}{\rho}\int \mathbf{n}\,(d\varepsilon)\,\mathbb{I}_{\varepsilon>0}\left(1-e^{-\rho(V(\varepsilon)\wedge D\wedge T_a(\varepsilon))}\right).
\end{aligned} \tag{4.9}$$

These expressions can be calculated using William's description of Itô's measure. Let us notice that, conditionally to the maximum of the excursion m, the law of the life length V is the sum of two independent hitting times $T_m(R)$ for a Bessel-3 process. Also, we have "in law"

$$T_a(\varepsilon)\wedge V(\varepsilon) = \left(T_m(R)+T_m(R')\right)\mathbb{I}_{m<a}+T_a(R)\,\mathbb{I}_{m\geq a}$$

for an independent Bessel-3 process R'. The law of $T_h(R)$ can be found for a Bessel-3 starting at zero; we just illustrate the approach. From Borodin and Salminen (1996, p. 339, formula 2.0.1) we get that for a Bessel-3 process R starting from a, with $h\geq a$,

$$\mathbb{E}_a\left[e^{-\frac{\alpha^2}{2}T_h(R)}\right]=\frac{h}{a}\frac{\sinh(\alpha a)}{\sinh(\alpha h)}.$$

To obtain the law for the process starting at zero, we take the limit of the Laplace Transform. We have

$$\mathbb{E}_0\left[e^{-\frac{\alpha^2}{2}T_h(R)}\right]=\frac{\alpha h}{\sinh(\alpha h)}.$$

This Laplace transform can be inverted (cf Biane and Yor (1987)) and gives

$$\mathbb{P}_0\left(T_h(R)\in dt\right)=\frac{\pi^2}{2h^2}\sum_{k=-\infty}^{+\infty}(-1)^{k+1}k^2e^{-\frac{k^2\pi^2t}{2h^2}}dt. \tag{4.10}$$

So we can write for 4.9

$$\begin{aligned}
&\frac{1}{\rho}\int \mathbf{n}\,(d\varepsilon)\,\mathbb{I}_{\varepsilon>0}\left(1-e^{-\rho(V(\varepsilon)\wedge D\wedge T_a(\varepsilon))}\right) \\
= &\frac{1}{2\rho}\int_0^a\frac{dm}{m^2}\int_0^\infty\left(1-e^{-\rho v\wedge D}\right)\mathbb{P}_0\left(T_m(R)+T_m(R')\in dv\right) \\
&+\frac{1}{2\rho}\int_a^\infty\frac{dm}{m^2}\int_0^\infty\left(1-e^{-\rho v\wedge D}\right)\mathbb{P}_0\left(T_a(R)\in dv\right) \\
= &\frac{1}{2\rho}\int_0^a\frac{dm}{m^2}\int_0^D\left(1-e^{-\rho v}\right)\mathbb{P}_0\left(T_m(R)+T_m(R')\in dv\right) \\
&+\frac{1}{2\rho}\int_0^a\frac{dm}{m^2}\int_D^\infty\left(1-e^{-\rho D}\right)\mathbb{P}_0\left(T_m(R)+T_m(R')\in dv\right) \\
&+\frac{1}{2\rho}\int_a^\infty\frac{dm}{m^2}\int_0^D\left(1-e^{-\rho v}\right)\mathbb{P}_0\left(T_a(R)\in dv\right) \\
&\frac{1}{2\rho}\int_a^\infty\frac{dm}{m^2}\int_D^\infty\left(1-e^{-\rho D}\right)\mathbb{P}_0\left(T_a(R)\in dv\right).
\end{aligned}$$

Using 4.10 we obtain:

$$\begin{aligned}
&\frac{1}{\rho}\int \mathbf{n}\left(d\varepsilon\right)\mathbb{I}_{\varepsilon_v>0}\left(1-e^{-\rho(V(\varepsilon)\wedge D\wedge T_a(\varepsilon))}\right)\\
=\;&\frac{1}{2\rho}\int_0^a\frac{dm}{m^2}\frac{\pi^4}{4m^4}\int_0^D du\left(1-e^{-\rho u}\right)\int_0^u dv\sum_{k\in\mathbb{Z}}(-1)^{k+1}k^2e^{-\frac{k^2\pi^2v}{2m^2}}\sum_{l\in\mathbb{Z}}(-1)^{l+1}l^2e^{-\frac{l^2\pi^2(u-v)}{2m^2}}\\
&+\frac{1}{2\rho}\left(1-e^{-\rho D}\right)\int_0^a\frac{dm}{m^2}\frac{\pi^4}{4m^4}\int_D^\infty du\int_0^u dv\sum_{k\in\mathbb{Z}}(-1)^{k+1}k^2e^{-\frac{k^2\pi^2v}{2m^2}}\sum_{l\in\mathbb{Z}}(-1)^{l+1}l^2e^{-\frac{l^2\pi^2(u-v)}{2m^2}}\\
&+\frac{1}{2a\rho}\int_0^D dv\left(1-e^{-\rho v}\right)\frac{\pi^2}{2a^2}\sum_{k\in\mathbb{Z}}(-1)^{k+1}k^2e^{-\frac{k^2\pi^2v}{2a^2}}\\
&+\frac{1}{2a\rho}\left(1-e^{-\rho D}\right)\int_D^\infty dv\frac{\pi^2}{2a^2}\sum_{k\in\mathbb{Z}}(-1)^{k+1}k^2e^{-\frac{k^2\pi^2v}{2a^2}}.
\end{aligned}$$

Now, for 4.8, we can use the same calculation, and we obtain

$$\begin{aligned}
&\int \mathbf{n}\left(d\varepsilon\right)\mathbb{I}_{\varepsilon>0}\left(1-e^{-\rho V(\varepsilon)}\mathbb{I}_{V(\varepsilon)<D}\mathbb{I}_{\sup_{0\le u\le V(\varepsilon)}\varepsilon_u<a}\right)\\
=\;&\int \mathbf{n}\left(d\varepsilon\right)\mathbb{I}_{\varepsilon>0}\mathbb{I}_{\sup_{0\le u\le V(\varepsilon)}\varepsilon_u<a}\left(1-e^{-\rho V(\varepsilon)}\mathbb{I}_{V(\varepsilon)<D}\right)\\
&+\int \mathbf{n}\left(d\varepsilon\right)\mathbb{I}_{\varepsilon>0}\mathbb{I}_{\sup_{0\le u\le V(\varepsilon)}\varepsilon_u>a}\\
=\;&\frac{1}{2}\int_0^a\frac{dm}{m^2}\left(1-\mathbb{E}_0^{(3)}\left[e^{-\rho(T_m+T'_m)}\mathbb{I}_{T_m+T'_m<D}\right]\right)+\frac{1}{2}\int_a^\infty\frac{dm}{m^2}\\
=\;&\frac{1}{2}\int_0^a\frac{dm}{m^2}\left(1-\frac{\pi^4}{4m^4}\int_0^D due^{-\rho u}\int_0^u dv\sum_{k\in\mathbb{Z}}(-1)^{k+1}k^2e^{-\frac{k^2\pi^2v}{2m^2}}\sum_{l\in\mathbb{Z}}(-1)^{l+1}l^2e^{-\frac{l^2\pi^2(u-v)}{2m^2}}\right)\\
&+\frac{1}{2a}.
\end{aligned}$$

As for the remaining terms:

$$\begin{aligned}
\int \mathbf{n}\left(d\varepsilon\right)\mathbb{I}_{\varepsilon\le0}\left(1-e^{-\rho V(\varepsilon)}\right)&=\frac{1}{2}\int_0^\infty\frac{dv}{\sqrt{2\pi v^3}}\left(1-e^{-\rho v}\right)\\
&=\sqrt{\frac{\rho}{2}},
\end{aligned}$$

and

$$\begin{aligned}
&\int \mathbf{n}\left(d\varepsilon\right)\int_0^{V(\varepsilon)}dve^{-\rho v}\mathbb{I}_{\varepsilon_v\le0}\\
=\;&\frac{1}{2\rho}\int \mathbf{n}\left(d\varepsilon\right)\left(1-e^{-\rho V(\varepsilon)}\right)\\
=\;&\frac{1}{2\rho}\int_0^\infty\frac{dv}{\sqrt{2\pi v^3}}\left(1-e^{-\rho v}\right)\\
=\;&\frac{1}{\sqrt{2\rho}}.
\end{aligned}$$

So, we can gather all the terms and write the following Laplace transform, using the previous Lemma:

$$\mathbb{E}\left[e^{-\rho\tau}\right]=1-\rho\frac{\int \mathbf{n}\left(d\varepsilon\right)\int_0^{V(\varepsilon)}dve^{-\rho v}\left(\mathbb{I}_{\varepsilon_v>0}\mathbb{I}_{v<D}\mathbb{I}_{\sup_{u\le v}\varepsilon_u\le a}+\mathbb{I}_{\varepsilon_v\le0}\right)}{\int \mathbf{n}\left(d\varepsilon\right)e^{-\rho V(\varepsilon)}\mathbb{I}_{\varepsilon>0}\left(\mathbb{I}_{\sup_{0\le u\le V(\varepsilon)}\varepsilon_u>a}+\mathbb{I}_{V(\varepsilon)>D}-\mathbb{I}_{\sup_{0\le u\le V(\varepsilon)}\varepsilon_u>a}\mathbb{I}_{V(\varepsilon)>D}\right)}.$$

Replacing the densities with their closed form expression gives the final result. ■

We can double-check the behavior of this expression as τ converges to T_a, by seeing what happens when D goes to infinity. We know that

$$\begin{aligned} & \mathbb{E}_0^{(3)}\left[e^{-\rho(T_m(R)+T_m(R'))\wedge D}\right] \\ = \; & \mathbb{E}_0^{(3)}\left[e^{-\rho(T_m(R)+T_m(R'))}\mathbb{I}_{T_m(R)+T_m(R')<D}+e^{-\rho D}\mathbb{I}_{T_m(R)+T_m(R')>D}\right] \end{aligned}$$

so

$$\lim_{D\to\infty}\mathbb{E}_0^{(3)}\left[e^{-\rho(T_m(R)+T_m(R'))\wedge D}\right]=\left(\mathbb{E}_0^{(3)}\left[e^{-T_{2m}(R)}\right]\right)^2=\frac{2\rho m^2}{\sinh^2\left(m\sqrt{2\rho}\right)},$$

and

$$\lim_{D\to\infty}\mathbb{E}_0^{(3)}\left[e^{-\rho T_a(R)\wedge D}\right]=\frac{a\sqrt{2\rho}}{\sinh\left(a\sqrt{2\rho}\right)}.$$

In addition,

$$\lim_{D\to\infty}\mathbb{E}_0^{(3)}\left[e^{-\rho(T_m(R)+T_m(R'))}\mathbb{I}_{T_m(R)+T_m(R')<D}\right]=\frac{2\rho m^2}{\sinh^2\left(m\sqrt{2\rho}\right)}.$$

We can write

$$\begin{aligned} & \int_0^a\frac{dm}{m^2}\left(1-\mathbb{E}_0^{(3)}\left[e^{-\rho(T_m+T'_m)}\right]\right) \\ = \; & \int_0^\infty\frac{dm}{m^2}\left(1-\mathbb{E}_0^{(3)}\left[e^{-\rho(T_m+T'_m)}\right]\right)-\int_a^\infty\frac{dm}{m^2}\left(1-\mathbb{E}_0^{(3)}\left[e^{-\rho(T_m+T'_m)}\right]\right) \\ = \; & \int_0^\infty\frac{dv}{\sqrt{2\pi v^3}}\left(1-e^{-\rho v}\right)-\frac{1}{a}+\frac{4\rho}{\sqrt{2\rho}\left(1-e^{2a\sqrt{2\rho}}\right)} \\ = \; & \sqrt{2\rho}-\frac{1}{a}+\frac{4\rho}{\sqrt{2\rho}\left(1-e^{2a\sqrt{2\rho}}\right)}. \end{aligned}$$

Thanks to a change of variable ($x=e^{\alpha z}$), it is easy to check that

$$\begin{aligned} \int_a^b\frac{dz}{\sinh^2(\alpha z)} &= \frac{2}{\alpha}\left(\frac{e^{2\alpha a}}{e^{2\alpha a}-1}-\frac{e^{2\alpha b}}{e^{2\alpha b}-1}\right) \text{ and} \\ \int_a^\infty\frac{dz}{\sinh^2(\alpha z)} &= \frac{2}{\alpha\left(e^{2\alpha a}-1\right)}. \end{aligned}$$

Also, let us notice that under Ito's measure, William's and Ito's description of the length of an excursion should agree. This is a quick way of showing that:

$$\begin{aligned} & \frac{1}{2}\int_0^\infty\frac{dv}{\sqrt{2\pi v^3}}\left(1-e^{-\rho v}\right) \\ = \; & \int\mathbf{n}(d\varepsilon)\,\mathbb{I}_{\varepsilon>0}\left(1-e^{-\rho V(\varepsilon)}\right) \\ = \; & \frac{1}{2}\int_0^\infty\frac{dm}{m^2}\int_0^\infty\mathbb{P}_0\left(T_m(R)+T_m(R')\in dv\right)\left(1-e^{-\rho v}\right) \\ = \; & \int_0^\infty\frac{dm}{2m^2}\left(1-\frac{2\rho m^2}{\sinh^2\left(m\sqrt{2\rho}\right)}\right). \end{aligned}$$

In consequence, we have

$$\begin{aligned}
&\lim_{D\to\infty} \mathbb{E}\left[e^{-\rho\tau}\right] \\
= &\ 1 - \frac{\int_0^a \frac{dm}{m^2}\left(1-\mathbb{E}_0^{(3)}\left[e^{-\rho(T_m+T'_m)}\right]\right)+\sqrt{2\rho}+\frac{1}{a}\left(1-\mathbb{E}_0^{(3)}\left[e^{-\rho T_a}\right]\right)}{\int_0^a \frac{dm}{m^2}\left(1-\mathbb{E}_0^{(3)}\left[e^{-\rho(T_m+T'_m)}\right]\right)+\frac{1}{a}+\sqrt{2\rho}} \\
= &\ \frac{\frac{1}{a}\mathbb{E}_0^{(3)}\left[e^{-\rho T_a}\right]}{\int_0^a \frac{dm}{m^2}\left(1-\mathbb{E}_0^{(3)}\left[e^{-\rho(T_m+T'_m)}\right]\right)+\frac{1}{a}+\sqrt{2\rho}} \\
= &\ \frac{\frac{\sqrt{2\rho}}{\sinh\left(a\sqrt{2\rho}\right)}}{2\sqrt{2\rho}+\frac{4\rho}{\sqrt{2\rho}\left(e^{2a\sqrt{2\rho}}-1\right)}} = \frac{\frac{1}{\sinh\left(a\sqrt{2\rho}\right)}}{2-\frac{2}{\left(1-e^{2a\sqrt{2\rho}}\right)}} = \frac{\left(1-e^{2a\sqrt{2\rho}}\right)}{-2e^{2a\sqrt{2\rho}}\sinh\left(a\sqrt{2\rho}\right)} \\
= &\ \frac{\left(1-e^{2a\sqrt{2\rho}}\right)}{e^{2a\sqrt{2\rho}}e^{-a\sqrt{2\rho}}\left(1-e^{2a\sqrt{2\rho}}\right)} = e^{-a\sqrt{2\rho}},
\end{aligned}$$

As could be expected.

We can also check how the expression behaves as $a \to \infty$. We can write directly:

$$\begin{aligned}
\lim_{a\to\infty} \mathbb{E}\left[e^{-\rho\tau}\right] &= 1 - \frac{\int_0^\infty \frac{dm}{m^2}\left(1-\mathbb{E}_0^{(3)}\left[e^{-\rho(T_m+T'_m)\wedge D}\right]\right)}{\int_0^\infty \frac{dm}{m^2}\left(1-\mathbb{E}_0^{(3)}\left[e^{-\rho(T_m+T'_m)}\mathbb{I}_{T_m+T'_m<D}\right]\right)+\sqrt{2\rho}} \\
&\quad - \frac{\int_0^\infty \frac{dv}{\sqrt{2\pi v^3}}\left(1-e^{-\rho v}\right)}{\int_0^\infty \frac{dm}{m^2}\left(1-\mathbb{E}_0^{(3)}\left[e^{-\rho(T_m+T'_m)}\mathbb{I}_{T_m+T'_m<D}\right]\right)+\sqrt{2\rho}}.
\end{aligned}$$

Now, using the agreement between Ito's description and William's, and by simplifying some, we have

$$\begin{aligned}
\lim_{a\to\infty} \mathbb{E}\left[e^{-\rho\tau}\right] &= \frac{\int_0^\infty \frac{dv}{\sqrt{2\pi v^3}}\left(1-e^{-\rho v}\mathbb{I}_{v<D}\right)+\sqrt{2\rho}-\int_0^\infty \frac{dv}{\sqrt{2\pi v^3}}\left(1-e^{-\rho v\wedge D}\right)-\sqrt{2\rho}}{\int_0^\infty \frac{dv}{\sqrt{2\pi v^3}}\left(1-e^{-\rho v}\mathbb{I}_{v<D}\right)+\sqrt{2\rho}} \\
&= \frac{\int_0^\infty \frac{dv}{\sqrt{2\pi v^3}}\left(1-e^{-\rho v}\mathbb{I}_{v<D}\right)-\int_0^\infty \frac{dv}{\sqrt{2\pi v^3}}\left(1-e^{-\rho v\wedge D}\right)}{\int_0^\infty \frac{dv}{\sqrt{2\pi v^3}}\left(1-e^{-\rho v}\mathbb{I}_{v<D}\right)+\sqrt{2\rho}} \\
&= \frac{e^{-\rho D}\int_D^\infty \frac{dv}{\sqrt{2\pi v^3}}}{\int_0^D \frac{dv}{\sqrt{2\pi v^3}}\left(1-e^{-\rho v}\right)+\int_D^\infty \frac{dv}{\sqrt{2\pi v^3}}+\sqrt{2\rho}} \\
&= \frac{\frac{2e^{-\rho D}}{\sqrt{2\pi D}}}{\sqrt{2\rho}+2\sqrt{2\rho}\left(\frac{e^{-\rho D}-1}{\sqrt{2\pi}\sqrt{2\rho D}}+\mathcal{N}\left(\sqrt{2\rho D}\right)-\frac{1}{2}\right)+\frac{1}{\sqrt{2\pi D}}} \\
&= \frac{\frac{2e^{-\rho D}}{\sqrt{2\pi D}}}{2\frac{e^{-\rho D}}{\sqrt{2\pi D}D}+2\sqrt{2\rho}\mathcal{N}\left(\sqrt{2\rho D}\right)} \\
&= \frac{1}{1+\sqrt{2\pi D}\rho e^{-\rho D}\mathcal{N}\left(\sqrt{2\rho D}\right)}.
\end{aligned}$$

This is the result from Chesney, Jeanblanc, Yor (1997).

Expression of the desired quantities

Using the preceding computations, we can write that for any positive ρ

$$\mathbb{E}\left[e^{-\rho T_a}\mathbb{I}_{H_D^+\geq T_a}\right] = \frac{1-\mathsf{AB}}{e^{a\sqrt{2\rho}}-\mathsf{A}}$$

where

$$\mathsf{A} = \frac{\sum_{k\in\mathbb{Z}}\int_0^{\frac{a}{\sqrt{D}}} dz\left(z+\frac{2ka}{\sqrt{D}}\right)\exp\left(\sqrt{2\rho D}z-\frac{1}{2}\left(z+\frac{2ka}{\sqrt{D}}\right)^2\right)}{\sum_{k\in\mathbb{Z}}\int_0^{\frac{a}{\sqrt{D}}} dz\left(z+\frac{2ka}{\sqrt{D}}\right)\exp\left(-\frac{1}{2}\left(z+\frac{2ka}{\sqrt{D}}\right)^2\right)} = \mathsf{A}_{\sqrt{2\rho}}$$

The other expression we are interested in can be rewritten, using the same independence results as earlier,

$$\begin{aligned}
&\mathbb{E}\left[e^{-\rho H_D^+}\mathbb{I}_{H_D^+\leq T_a}f\left(B_{H_D^+}\right)\right]\\
=\ &\mathbb{E}\left[e^{-\rho H_D^+}\mathbb{I}_{\sup_{0\leq t\leq g\left(H_D^+\right)} B_t\leq a}\right]\mathbb{E}\left[f\left(B_{H_D^+}\right)\mathbb{I}_{\sup_{g\left(H_D^+\right)\leq t\leq H_D^+} B_t\leq a}\right]\\
=\ &\mathbb{E}\left[e^{-\rho H_D^+}\mathbb{I}_{H_D^+\leq T_a}\right]\sum_{k\in\mathbb{Z}}\int_0^{\frac{a}{\sqrt{D}}} dz f\left(z\sqrt{D}\right)\left(z+\frac{2ka}{\sqrt{D}}\right)\exp\left(-\frac{1}{2}\left(z+\frac{2ka}{\sqrt{D}}\right)^2\right)\\
=\ &\mathbb{E}\left[e^{-\rho H_D^+}\mathbb{I}_{H_D^+\leq T_a}\right]\mathsf{K}(f)
\end{aligned}$$

Now, we know that

$$\begin{aligned}
\mathbb{E}\left[e^{-\rho\tau}\right] &= \mathbb{E}\left[e^{-\rho T_a}\mathbb{I}_{H_D^+\geq T_a}\right]+\mathbb{E}\left[e^{-\rho H_D^+}\mathbb{I}_{H_D^+\leq T_a}\right]\\
\text{That is, } \mathbb{E}\left[e^{-\rho H_D^+}\mathbb{I}_{H_D^+\leq T_a}\right] &= \mathsf{B}-\frac{1-\mathsf{AB}}{e^{a\sqrt{2\rho}}-\mathsf{A}}\\
&= \frac{e^{a\sqrt{2\rho}}\mathsf{B}-1}{e^{a\sqrt{2\rho}}-\mathsf{A}}.
\end{aligned}$$

So

$$\begin{aligned}
&\mathbb{E}\left[e^{-\rho H_D^+}\mathbb{I}_{H_D^+\leq T_a}f\left(B_{H_D^+}\right)\right]\\
=\ &\sum_{k\in\mathbb{Z}}\int_0^{\frac{a}{\sqrt{D}}} dz f\left(z\sqrt{D}\right)\left(z+\frac{2ka}{\sqrt{D}}\right)\exp\left(-\frac{1}{2}\left(z+\frac{2ka}{\sqrt{D}}\right)^2\right)\frac{e^{a\sqrt{2\rho}}\mathsf{B}-1}{e^{a\sqrt{2\rho}}-\mathsf{A}}.
\end{aligned}$$

The proof of the theorem is now complete. ■

The large company's strategic behavior

In the first section the strategic behavior of the small company was presented. In the model, the small company knows exactly the investment level of the big company when the latter acts as a monopolist and can therefore preempt it in all cases.

We will consider in this section the case when the large company has a better information. First, we will examine the situation where both companies are fully informed. Then, we will consider on the contrary that the big company acts strategically on the basis of an imperfect information concerning the small one. Indeed, only the probability distribution of the small company investment level could be known.

If both firms have full information

Since both companies have advantages and disadvantages, there should be no systematic preemption as it was the case in the preceding section.

The big company knows that the small company knows its monopolist optimal investment level and its investment delay and that the latter firm will try to preempt. The big company will therefore try to decrease its investment level in order to avoid to be preempted. However, the small company will also decrease its investment level in order to keep preempting the big one as long as possible. At the limit, the small company won't invest bellow its minimum cost level K_2, which is higher than the costs of the big one and also known by the latter firm. Therefore, the small company will invest at its NPV level, and the big one will have an optimal level strictly higher than its NPV level, but smaller than the minimal cost level of the small firm and therefore than the monopolist optimal level.

The advantage of the small company consists of zero investment delay, the advantage of the big one in smaller costs. Following the approach explained in the first Section, we know the value of the big firm is given by:

$$V_1^*(S_0) = \sup_{L_1} \mathbb{E}\left(\exp\left(-rH^+_{L_1,D}\right)\left(S_{H^+_{L_1,D}} - K_1\right)\mathbb{I}_{H^+_{L_1,D}<T_{K_2}}\right)$$

$V_1(x)$ is a perpetual American up and out Barrier Call Option value, with a Parisian exercise boundary.. In other words, when at the level L_1 , below the barrier K_2, the decision to eventually exercise later is taken, the payoff is not the intrinsic value $L_1 - K_1$. It is obtained after a lag D, only if the underlying value has remained above L_1 (and below the barrier K_2) during a period of D units of time.

Proposition 17 *The value of the large firm if it knows the constraints of its competitors is (keeping the notations defined in the first section) the supremum over L_1 of*

$$\begin{aligned} V_1\left(S_0\right) &= \mathbb{E}_{S_0}\left[\exp\left(-rH^+_{L_1,D}\right)\left(S_{H^+_{L_1,D}} - K\right)\mathbb{I}_{T_{K_2}\geqslant H^+_{L_1,D}}\right] \\ &= \left(\frac{L_1}{S_0}\right)^{\sqrt{\frac{2r+m^2}{2\sigma^2}}} \frac{\left(e^{a\sqrt{2\rho}}\mathsf{B}-1\right)\mathsf{K}\left(f\right)}{e^{a\sqrt{2\rho}}-\mathsf{A}} \end{aligned}$$

Proof. Using the theorem shown in Section 2, we can give a closed form expression for the value. Indeed, we have

$$\begin{aligned} & \mathbb{E}\left[e^{-\rho H^+_D}\mathbb{I}_{H^+_D\leq T_a} f\left(W_{H^+_D}\right)\right] \\ = & \ \mathsf{K}\left(f\right)\frac{e^{a\sqrt{2\rho}}B-1}{e^{a\sqrt{2\rho}}-A} \end{aligned}$$

If we apply a change of measure and after a few substitutions, we obtain the proposition's result. ■

At the optimum, the first order condition gives the level at which the large firm will trigger its decision. Note that in all cases, the value of the large firm combined with the value of the small firm should equal the total value of the small firm when it can preempt systematically. This can be clearly seen in the formal expression of $V_1\left(S_0\right)$ and $V\left(S_0\right)$ (as given in the first section).

If we do not assume that the small firm realizes a null profit (by investing at its break-even point), then the strategies of both firms must be optimal in an equilibrium.

$$\begin{cases} V_1(S_0, L_2) = \sup_{L_1} \mathbb{E}\left(\exp\left(-rH^+_{L_1,D}\right)\left(S_{H^+_{L_1,D}} - K_1\right)\mathbb{I}_{H^+_{L_1,D}<T_{L_2}}\right) \\ V_2(S_0, L_1) = \sup_{L_2} \mathbb{E}\left(\exp\left(-rT_{L_2}\right)\left(L_2 - K_2\right)\mathbb{I}_{H^+_{L_1,D}>T_{L_2}}\right) \end{cases}$$

or

$$\begin{cases} f_1\left(L_2\right) = \arg\max_{L_1}\left\{\mathbb{E}\left(\exp\left(-rH^+_{L_1,D}\right)\left(S_{H^+_{L_1,D}} - K_1\right)\mathbb{I}_{H^+_{L_1,D}<T_{f_2(L_1)}}\right)\right\} \\ f_2\left(L_1\right) = \arg\max_{L_2}\left\{\mathbb{E}\left(\exp\left(-rT_{L_2}\right)\left(L_2 - K_2\right)\mathbb{I}_{H^+_{f_1(L_2),D}>T_{L_2}}\right)\right\} \end{cases}$$

These expressions can be written explicitly, thanks to the theorem of Section 2, but the resolution of the system cannot be done explicitly.

If the big firm has limited information

In this section, we study more particularly the behavior of the large firm when it is aware there is a smaller firm that can potentially preempt its investment opportunity, but it does not know its constraints. Information is still asymmetrical in that case, but we allow the big firm to follow a strategic behavior.. It is natural to consider that the large firm has only access to aggregate information regarding its competitor: industry surveys very often would give industry averages or distributions that are relevant to the big firm's analysis, but no precise and individual information. So the large firm has to assume the characteristics of its opponent are drawn from a known distribution, but their value is known only to the small firm.

This "distributional strategies approach" has been first developed by Milgrom and Weber (1985), and then applied to real options by Lambrecht and Perraudin (1994, 1996), who consider two similar firms with no delay constraint. We propose here a simple and rational explanation for the use of an *a-priori* distribution for the decision parameters of the large firm. In a specific industry, managers are able to produce subjective distributions for the level of profitability at which competitors would enter a market. This comes from industry wide surveys, but also from the analysis they perform on their own company.

We assume that for the large company, the level at which the small company will invest is a random variable independent from the source of randomness we have been considering so far, and we write its density $\mathbb{P}\left(\widehat{L_2} \in dh\right)$. This is valid at the outset of the problem, that is when the large firm discovers it has an investment opportunity, or when it discovers it has a competitor, if it has not invested yet. Implicitly, we assume the large company is aware of the shape of the constraints incurred by the small firm: it is making the hypothesis that the investment trigger is a hitting time.

Finally, the large firm uses the available information, that is the filtration generated by the variable process, to update its subjective distributional assumption over the value of its opponent trigger. It conditions its distribution assumption by the fact the opponent has not invested yet. All the information the large firm has, indeed, is whether the stopping time that represents its opponent's threshold has been realized or not.

To the large firm, the "strategic" value of the project is therefore

$$V_1(S_t) = \sup_{L_1} \mathbb{E}_{S_t}\left[e^{-rH^+_{D,L_1}} \left(S_{H^+_{D,L_1}} - K_1 \right) \mathbb{I}_{T_{\widehat{L_2}} > H^+_{D,L_1}} \middle| M_t \le \widehat{L_2} \right]$$

where M is the running supremum of the process S. We have $M_t = \sup_{0 \le s \le t} S_s$ where 0 is the instant since when the large firm believes the small firm has contemplated investing. Here, on the contrary to the usual situation with these notations, the random variable is $\widehat{L_2}$ and the parameter is M_t, as the latter is $\mathcal{F}_t$-measurable.

We obtain immediately

$$\begin{aligned} V_1(S_t) &= \int_{M_t}^{+\infty} \frac{\mathbb{P}\left(\widehat{L_2} \in dh\right)}{\mathbb{P}\left(M_t \le \widehat{L_2}\right)} \mathbb{E}_{S_t}\left[e^{-rH^+_{D,L_1}} \left(S_{H^+_{D,L_1}} - K_1 \right) \mathbb{I}_{T_h > H^+_{D,L_1}} \right] \\ &= \int_{M_t}^{+\infty} \frac{\mathbb{P}\left(\widehat{L_2} \in dh\right)}{\mathbb{P}\left(M_t \le \widehat{L_2}\right)} \left(\frac{L_1}{S_t} \right)^{\sqrt{\frac{2r+m^2}{2\sigma^2}}} \frac{\left(e^{a\sqrt{2\rho}} \mathsf{B} - 1 \right) \mathsf{K}(f)}{e^{a\sqrt{2\rho}} - \mathsf{A}} \end{aligned}$$

With time, the large firm changes its anticipation of $\widehat{L_2}$, as the running maximum increases, it changes its own threshold. The optimal investment threshold for the large firm will therefore evolve and reflect its original subjective distribution, the realized path of the observed variable, and the fact the smaller firm has not invested so far.

However, since the small firm knows the characteristics of the large firm, it can take into account in its strategy the fact that the large firm will alter its strategy. If the large firm makes its decision thresholds path-dependent as we have described above, then the small firm will take it into account. In such a situation, the optimal strategy followed by the small firm becomes more complex to describe and model.

Concluding remarks

We have seen how large entities and smaller entities face different constraints when they contemplate investing in a project. The value of these investments is related to the value of certain options, when the investors are each in a monopolistic situation. In a competition situation where the first to invest preempts totally the project, we have exposed how a particular class of options, Parisian American options, allows to model the combined constraints faced by investors. The largest firm has an option to invest which can be exercised only under a given barrier (over which the smaller firms invests immediately) and according to its investment delay constraints.

We have proposed a pricing formula for these options which is decomposed into its value if exercised at the upper barrier, and its value if it is exercised at the Parisian time. To calculate the result, we have given a new result pertaining to the first instant a Brownian Motion hits a level or spends more than a given amount of time above a lower level.

The pricing formulas can be applied to specific competition situations which exhibit a disbalance between the characteristics of the opponents. From the technical viewpoint, the same approach helps value options where functionals of excursions intervene (such as the first instant when the area of an excursion reaches a certain level).

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An application to delayed investment decision

In this section we describe an application of the main theorem in Chapter 4 to monopolistic investment decisions with a delay. As we explained in Chapter 2, a Parisian stopping time can be used to account for an investment strategy under a delay constraint. Indeed, it is more optimal to follow the Parisian rule, rather than "blindly" invest D units of time after a threshold has been reached by the underlying variable.

This framework does not take into account the fact that, in reality, it may be possible to speed things up at a cost: for a (significantly) higher entry cost, the project could be started straight away. For example, if the delay is associated to a financing constraint, it is certainly possible to issue very cheap capital, or borrow money at a higher rate, and obtain the necessary funds immediately. In this case, the cost for immediacy translates into higher entry costs.

We note

- K_i the cost associated to an immediate start of the project,
- K_e the cost of entry after a delay of D,
- L_1 the level above which the underlying variable has to stay for D units of time to trigger the investment at a cost of K_e,
- and L the level that will trigger an immediate investment at the cost of K_i.

Proposition 18 *The value of an investment project, with barriers $L_1 \leq L$ and $K_e < K_i$ is equal to*

$$\begin{aligned} V(S_0) &= \mathbb{E}\left[\exp(-rT_L)(L-K)\mathbb{I}_{T_L<H^+_{L_1,D}} + \exp\left(-rH^+_{L_1,D}\right)\left(S_{H^+_{L_1,D}} - K\right)\mathbb{I}_{T_L\geqslant H^+_{L_1,D}}\right] \\ &= \left(\frac{L_1}{S_0}\right)^{\sqrt{\frac{2r+m^2}{2\sigma^2}}} \frac{\left[(1-\mathsf{AB})f_i(a) + \left(e^{a\sqrt{2\rho}}\mathsf{B}-1\right)\frac{\mathsf{K}(f_e)}{\mathsf{K}(1)}\right]}{e^{a\sqrt{2\rho}}-\mathsf{A}} \end{aligned}$$

with $\rho = r + \frac{m^2}{2}$, $a = \frac{\ln\left(\frac{L}{L_1}\right)}{\sigma}$, $f_i(z) = \frac{1}{\delta}e^{mz}(L_1e^{\sigma z} - K_i)$, $f_e(z) = \frac{1}{\delta}e^{mz}(L_1e^{\sigma z} - K_e)$ *and* A, B, K *defined in Theorem 13.*

The optimal value of the investment is $\sup_{L,L_1} V(S_0)$.

A decomposition appears between the part of the price that is due to an immediate entry at a higher cost,

$$\left(\frac{L_1}{S_0}\right)^{\sqrt{\frac{2r+m^2}{2\sigma^2}}} \frac{(1-\mathsf{AB})\, f_i\,(a)}{e^{a\sqrt{2\rho}}-\mathsf{A}}$$

and the part of the price linked to a an entry at the lower cost of K_e, but after a delay of D.

$$\left(\frac{L_1}{S_0}\right)^{\sqrt{\frac{2r+m^2}{2\sigma^2}}} \frac{\left(e^{a\sqrt{2\rho}}\mathsf{B}-1\right)\frac{\mathsf{K}(f_e)}{\mathsf{K}(1)}}{e^{a\sqrt{2\rho}}-\mathsf{A}}.$$

Chapter 5 NOISY INFORMATION AND INVESTMENT DECISION[1]

We have seen in the previous chapters that firms managers have an option to invest and should not invest as soon as it gets in the money. Real options are not only academic thought exercises, they are more and more used by corporate decision-makers, more or less consciously. As Luehrman puts it in the Harvard Business Review (1998):

> The analogy between financial options and corporate investments that create future opportunities is both intuitively appealing and increasingly well accepted. Executives readily see why investing today in R&D, or in a new marketing program, or even in certain capital expenditures (a phased plant expansion, say) can generate the possibility of new products or new markets tomorrow.

When we look at empirical evidence concerning these theories, the option premium detected in those models seems to have a great statistical significance. However, most tests find that the option premium generated by the data is generally spread over and under the value generated by the models [2].

In competitive markets the winner's curse can account for the undervaluation associated with standard real option models. Another reason for this undervaluation may be that many models developed so far are too simple to account for the investment projects' embedded options. Managers can make choices about the project's characteristics and this flexibility creates embedded options that add value to the project.

One other major characteristic of the capital budgeting process is the noise existing in the information available to the investor. Almost without exception, the literature on real options has assumed that managers have perfect information concerning the decision variable underlying their investment decisions. Nevertheless, as noticed by Williams (1995), since most markets for real assets are decentralized, the information available to the investor is generally noisy. Recently, Decamps and Mariotti (2000) have looked at the effect of imperfect information in a duopolistic competition case. They show that having to acquire information creates further incentives to delay investment.

The present paper shows that the noise in the information available to the investor can account for the overvaluation of real options models. We use the computation of first passage times to derive closed-form formulas relating the value of the investment opportunity to the noisy decision variable. Our setting for the description of the noise is simple but brings out the generality of the idea. We show that noisy information tends to raise the first best investment boundary. As a consequence, the investment spending is depressed and the values of investment opportunities are reduced in comparison with the perfect forecast case.

Section one presents the model. Simulation results are reported in section two. Section three concludes the paper and discusses extensions.

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[1] A VERSION OF THIS CHAPTER, CO-WRITTEN WITH ERWAN MORELLEC, HAS BEEN PUBLISHED UNDER THE TITLE "NOISY INFORMATION AND INVESTMENT DECISION: A NOTE", *FINANCE* (PUF) 1999, 20(2).

[2] This phenomenon is documented empirically in Quigg (1993).

The model

In this section we develop a model where a firm has the opportunity to invest in a project yielding a stochastic continuous stream of cash flows. Throughout the analysis, capital markets are perfect with no transaction costs. Agents are risk neutral[3]and may lend and borrow freely at a constant instantaneous riskless rate r.

The noisy model

Consider a firm having the opportunity to invest in a new market. Once installed, this firm produces output with a constant capital stock k and variable factors of production. Uncertainty is represented by the demand shift parameter for the good produced by the firm. We consider that the observed variable is generic and of limited use to the firm. For example, it could be a national level of demand, estimated and published by a trade association, while the firm's market is in a specific region, or in a specific sub-class of product. With respect to the observed variable (the national level), the more specific information the firm is interested in is to be considered like noise: we assume it can not be estimated per se, but still some information can be gleaned about its properties. By conducting market pilots or market tests, the firm can measure how its own market behaves with respect to the national market. The firm can statistically estimate the mean and the dispersion of the noise. However, these market tests cannot be performed on a continuous basis, due to their cost. Consequently, the actual noise, as the discrepancy between the national level of demand and the local level of demand in our example, is unknown. The only variable on which the firm can rely for its investment decision is the observed national demand; for a given level of this national demand, the firm only knows the distribution of the local demand.

For the remainder of the paper, the observed or estimated demand shift parameter $(x_t, t \geq 0)$ is ruled by the geometric Brownian motion

$$x_T = x_t \exp\left\{\left(\mu - \sigma^2/2\right)(T-t) + \sigma\left(Z_T - Z_t\right)\right\} \geq 0 \tag{5.1}$$

where μ and σ are constant parameters and $(Z_t, t \geq 0)$ is a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

In a model with sequential search for investment opportunities, Williams (1995) shows that if the equilibrium value of an asset is a geometric Brownian motion then the average transaction price is ruled by a noisy geometric Brownian motion. In this paper we develop the ideas in the simplest possible context. Since we do not model the decision to acquire information to reduce noise, the error between the observed state variable and reality should not increase in norm over time. This means that the process modelling noise has to possess a limit distribution at infinite times. Therefore, noise cannot be conditioned by the level of the index x.

A practical way is to use the paradigm of mean-reversion, an Ornstein-Uhlenbeck process. We shall thus model noise as the exponential of the solution of

$$dy_t = \alpha\left(\kappa - y_t\right)dt + \theta dW_t,$$

[3]When agents are risk averse, we can operate a change of probability measure using Cameron-Martin-Girsanov theorem to develop the analysis in a risk neutral economy (see Harrison and Pliska (1981)). This approach which relies on the dynamic completeness of financial markets is used by He and Pindyck (1992) in a model where firms are characterized by a technology similar to ours.

where both processes Z and W are independent Brownian motions, that is $\langle Z, W \rangle_t = 0$.

We can now define the real process for the demand shift parameter as

$$\begin{aligned} a_t &= x_t e^{y_t} \\ &= x_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma Z_t + \kappa\left(1 - e^{-\alpha t}\right) + e^{-\alpha t} y_0 + \theta \int_0^t e^{-\alpha(t-u)} dW_u} \end{aligned}$$

that is the exponential of a sum of a drifted Brownian Motion and an Ornstein-Uhlenbeck process. This sum is a Gaussian process. y_t is normally distributed with mean and variance

$$\begin{aligned} \mathbb{E}(y_t) &= \kappa\left(1 - e^{-\alpha t}\right) + e^{-\alpha t}\mathbb{E}(y_0), \\ \mathbb{V}ar(y_t) &= e^{-2\alpha t}\mathbb{V}ar(y_0) + \frac{\theta^2}{2\alpha}\left(1 - e^{-2\alpha t}\right). \end{aligned}$$

Finally, if we want the law of y_t to be constant over time, y_0 has to be a Gaussian with mean κ and variance $\frac{\theta^2}{2\alpha}$ independent of W and Z.

Optimal investment decisions with noisy uncertainty

We determine in this section both the value of the investment opportunity and the level of the demand shift parameter that triggers investment. Hereafter we consider that the project, once installed, goes on producing the output flow forever, i.e. investment is irreversible[4]. Moreover, we assume that the technology of the project is such that the instantaneous profit function of an active firm is given by

$$f(a_t, k_t) = \psi a_t^\gamma k_t^{1-\gamma},$$

where $\gamma \in [0, 1[$ and ψ is a scale parameter. This specification approximates the case of a pure equity firm with a Cobb-Douglas production function and facing an isoelastic demand curve. k is the production capacity, which we will assume constant so that $f(a_t, k_t) = f(a_t)$ and $k_t^{1-\gamma}$ can be "integrated" in ψ. Such a specification has already been used for the study of investment decisions under uncertainty by He and Pindyck (1992) and Abel and Eberly (1996). Notice that according to equation 5.1, we must have $r > \mu\gamma + \gamma(\gamma - 1)\sigma^2/2$ for the expected present value of operating profits to be finite.

As mentioned earlier, the investor cannot observe the real path of this state variable. Therefore, the firm acts as if the demand shock was ruled by the geometric Brownian motion $(x_t, t \geq 0)$, whereas it is ruled by $(a_t, t \geq 0)$. We assume however that the investor knows the first two moments of the distribution of the noise. If we denote $F(a_t, \infty)$ the expected present value of future profits when the investment is realized at time t, we have

$$F(a_t, \infty) = \int_t^\infty ds e^{-r(s-t)} \mathbb{E}_{x_t}[f(a_s)] = \Delta x_t^\gamma,$$

with

$$\Delta = \frac{\psi e^{\gamma\left(\kappa + \gamma\theta^2/4\alpha\right)}}{r - \gamma\mu + \sigma^2\gamma(\gamma - 1)/2}.$$

[4] Abel, Dixit, Eberly, and Pindyck (1996) note that "irreversibility may be important in practice because of "lemons effects" and because of capital specificity". The irreversibility assumption is realistic at least for economic activities which are highly capital intensive such as mining projects or offshore petroleum leases. Indeed for such activities, it is unusual to observe temporary shut down or capacity reduction.

The Markovian features of the model and the stationarity of the distribution of the payoffs generated by the active project imply that investment occurs at the first instant when the demand shock hits some constant threshold. Let us denote by $T_h(x)$ the first passage time of x at the level h. It is defined by

$$T_h(x) = \inf\{s \geq 0,\ x_s = h\}$$

If we denote C the entry cost, the value of investment opportunity for a given investment threshold h, $h > x_0$, is given by

$$V(x_0, h) = \mathbb{E}_{x_0}\left[\int_0^\infty dte^{-rt} f(a_t)\, \mathbb{I}_{t \geq T_h(x)}\right] - C\mathbb{E}_{x_0}\left[e^{-rT_h(x)}\right] \tag{5.2}$$

In this equation, the first term of the right hand side is the present value of expected profits generated by the investment project. The second term is the capital expenditure discounted between the expected investment time and the current date.

Using the strong property of Brownian motion, we can write equation 5.2 as

$$V(x_0, h) = \mathbb{E}_{x_0}\left[e^{-rT_h(x)}\left(F\left(a_{T_h(x)}, \infty\right) - C\right)\right].$$

Finally, using standard results concerning first passage times, we get the value of the option to invest and the demand threshold triggering investment as

$$V(x_0, h) = \left(\frac{x_0}{h}\right)^{\xi}\left[\frac{\psi h^{\gamma} \exp\left(\gamma\left(\kappa + \gamma\theta^2/4\alpha\right)\right)}{r - \gamma\mu + \sigma^2\gamma(\gamma - 1)/2} - C\right], \tag{5.3}$$

and

$$h^* = \exp\left(-\gamma\left(\kappa + \frac{\gamma\theta^2}{4\alpha}\right)\right)\left(\frac{\xi\left[r - \gamma\mu + \sigma^2\gamma(\gamma-1)/2\right]}{(\xi - \gamma)\psi}C\right)^{1/\gamma}, \tag{5.4}$$

where

$$\xi = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\frac{2r}{\sigma^2} + \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2}.$$

In equation 5.3, the first term of the right hand side represents the value of one dollar contingent on investment. The terms in the bracket account for the present value of the payoffs generated by the project. Finally, the first term of the right hand side in equation 5.4 accounts for the impact of the noise on the decision to invest.

When the firm decides to enter in the market the profit flow is

$$f(a_t, k) = \psi a_t^{\gamma}$$

Since the direct investment cost is C, equation 5.3 shows that it is optimal to invest when the expected present value of the profit flow generated by an active project exceeds its cost by the multiple $\frac{\xi}{\xi - \gamma} > 1$. This term is the usual option value parameter representing the value of waiting to invest. This coefficient makes the optimal investment threshold differ from the traditional Jorgensonian user cost of capital.

θ	$\rho(0.7,\theta,0.05)$
0	1
0.02	1.0004
0.04	1.0017
0.06	1.0038
0.08	1.0067
0.10	1.0106
0.12	1.0152
0.14	1.0208
0.16	1.0272
0.18	1.0346
0.20	1.0429

Table 5.1 Effect of Noise on Valuation

Noisy information and real options values

Using a sample of 2700 land transactions in Seattle[5] , Quigg (1993) shows that real options models perform better than standard Neoclassical models in explaining the real investment process. However, the coefficient corresponding to the increase in statistical significance of the option premium parameter in her regressions lies in most subsamples between 0.5 and 1.3 instead of the unit value associated with the empirical success of the models.

The undervaluation of the option model is often associated with the winner's curse in competitive markets[6]or with the value of the embedded options which, most of the time, are not taken into consideration. We show in this section that the noise in the information available to the investor can account for the overvaluation associated with real options models. By noise in this context, we specifically mean the uncertainty around the real level of the unobserved variable, rather than the fact that the variable evolves randomly.

Let us assume without loss of generality that the noise is centered around the decision process, i.e. $\kappa = -\frac{\theta^2}{4\alpha}$. The first factor of the right hand side of equation 5.4 can then be written as

$$\rho(\gamma,\theta,\alpha) = \exp\left\{\gamma(1-\gamma)\frac{\theta^2}{4\alpha}\right\} > 1$$

Table 5.1 on p. 83 represents the value of this factor, as a function of $\theta \in [0, 0.2]$ for $\gamma = 0.7$ and $\alpha = 0.05$.

One can observe that when information is noisy, investors choose an investment threshold higher than in standard real options model. Thus, the value of waiting to invest is higher and noisy information tends to depress investment. The ratio of the value of the investment opportunity with noisy information to that associated to the perfect forecast case is given by

$$r = \exp\left\{\xi(\gamma-1)\theta^2/4\alpha\right\}$$

[5] The building industry is characterized by many non-linearities for its profit and cost functions. For example, there exists a concave relationship between price and building size. Quigg points out that "in the market for commercial space, there might be a downward sloping demand curve for a given location, and it is likely that as the building size grows, the prime rentable space decreases as the proportion of the total space (e.g. more interior offices). In the market for residential space, doubling an appartment's size does not normally double the rent [...]."

[6] See Paddock, Siegel, and Smith (1988) for an example illustrating this point. In a nutshell, the "winner's curse" consists of the following reasoning: if someone wins a bidding contest, for example at an auction, then by definition he will have paid the highest price. This price is higher than the average of all the other bids, which can be taken as an indication of the real value of the good in question.

θ and σ	0.2	0.3	0.4
0.1	.9675	.97428	.97775
0.15	.92836	.94306	.95064
0.2	.87621	.90102	.91393

Table 5.2 Combined Effect of Noise and Volatility

$$= \exp\left\{\left(\frac{1}{2}-\frac{\mu}{\sigma^2}+\sqrt{\frac{2r}{\sigma^2}+\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)^2}\right)(\gamma-1)\,\theta^2/4\alpha\right\}.$$

Using the fact that $r > \mu\gamma+\gamma(\gamma-1)\,\sigma^2/2$, we know that $\left(\frac{1}{2}-\frac{\mu}{\sigma^2}+\sqrt{\frac{2r}{\sigma^2}+\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)^2}\right)$ is positive and the ratio r is decreasing as a function of θ and increasing as a function of α. This ratio tels us the value of the investment opportunity if there is noisy information (with an optimal investment threshold in this case) relative to the value of the investment opportunity if there is no noise (with a different investment threshold, optimal in that case). Let us look at the value of this ratio as a function of $\theta \in [0, 0.2]$ and $\sigma \in [0.15, 0.3]$ for $r = 0.075$, $\mu = 0.01$, $\gamma = 0.7$ and $\alpha = 0.05$.

Table 5.2 on p. 84 clearly shows how a greater noise in the estimation of the unobserved variable's position from the observed variable reduces the value of the investment opportunity. If for example the quality of the market tests conducted by a firm planning to invest in a new line of production deteriorated, the expected value of the investment project would decrease quite significantly, as shown in the numbers above. We have not addressed in our approach the issue of the cost of information (see Descamps and Mariotti, 2000), as we considered that the quality of available information is exogeneous for the firm. A simple rule to assess the marginal interest in buying a better information could be simply based on the increase in the project's value due to a lower θ.

Our results indicate that noise in the information available to managers can account for the fact that standard real options models overvalue investment opportunities. Indeed, we can see that the ratio lies between 0.8 and 1 for reasonable input parameter values. Moreover, the lower the risk of the firm's activity, the larger this value reduction. With a higher risk (larger σ), the variations of the observed state variable will so big they will tend to offset, at least partly, the effects of the noise on firm value.

Concluding remarks

The simple model we develop in this chapter shows that the noise in the information available to investors can explain part of their investment behavior. Indeed, the value of the investment opportunity can be reduced significantly for reasonable parameter values. The results in this chapter can be extended in several directions. First of all, a non-constant distribution of the noise could allow us to model more explicitly the acquisition of information by the investors. Second, strategic aspects can be introduced by considering competition between several investors looking for limited entry rights on a new market characterized by noisy information. In the case of limited entry, the "quota" could induce suboptimal rent runs with early entry due to dumping as in Bartolini (1995). On the other hand, noisy information could depress investment as the first mover would reveal the true state of nature to other investors.

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Chapter 6 HEDGING REAL OPTIONS WITH TRANSACTION COSTS: A CONVERGENCE RESULT[1]

For many firms, especially in the mining, oil, or commodities industries, the value of investment projects can be determined with real option theory. In these cases, an important argument underlying the valuation of projects is that the business risk, being linked to a traded product, can be hedged. Real options in that case can be considered as equivalent to complex options written on a commodity. Most large firms trading a commodity implement sophisticated hedging strategies to eliminate the market risk and concentrate on the business risk where they are the most efficient (oil companies can focus on oil exploration rather than running an exposure to oil markets, mining companies can focus on devising the most efficient ways of mineral extraction rather than playing the gold market...). When they implement hedging strategies for their market risks, commodities firms often face large transaction costs. Indeed, commodities are often traded the most heavily by investment banks (these banks benefit from a large capital basis they can leverage on commodities markets as well as in pure financial markets), which charge fees or transaction costs for the liquidity they provide. Because of these transaction costs, the valuation of real options should be altered.

In this chapter, we address the issue of hedging real options, and more generally hedging complex options, using an optimal combination of the underlying product and other derivatives written on it. The optimal hedging basket should minimize transaction costs. There is a trade-off between hedging with a derivative that replicates locally well the real option but with a high transaction cost, or with the underlying at a lower transaction cost but a higher frequency of rehedging.

There have been numerous approaches to study the pricing and hedging of derivative products in a continuous setting when there are transaction costs. The way these costs are modelled greatly conditions the tractability of the results. Indeed, as soon as transaction costs are considered to be proportional and not negligeable, a utility function must be specified for the trader, and the price verifies a free-boundary problem, the dimension of which is necessarily more than 3. Results can only be obtained numerically, as it is done in Davis et Al.. (1993), even in the simplest setting: opportunistic utility function, geometric Brownian Motion to model the underlying's price, European path-independent payoff, hedging strategy restrained to using the underlying.

In a series of papers, Hoggard et Al.. (1994), Whalley and Wilmott (1994 and 1997), efficient approximations for this kind of model have been developed. These approximations are based on asymptotic analysis, and suppose that transaction costs as well as the time delay between two portfolio rehedgings converge to zero. They write, after rather tedious calculations, the price of a plain-vanilla option, as the solution to an Integro-Partial Differential Equation, and propose numerical solutions. They also provide the limit hedging strategy for an arbitrary transaction cost structure. This model has not been extended to the path-dependent case.

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[1] A SHORTER VERSION OF THIS CHAPTER HAS BEEN PUBLISHED UNDER THE TITLE "A TRANSACTION COST CONVERGENCE RESULT FOR GENERAL HEDGING STRATEGIES" IN *STOCHASTIC MODELS* (DEKKER), 17(3), PP. 313-339 (2001).

In Hoggard et Al.. (1994), in Dewynne et Al.. (1995), and in Henrotte (1994), the particular case of proportional transaction costs going to zero as hedging frequency goes to infinity has been studied, along the lines of the seminal paper of Leland (1985). This model allows a perfect replication of the payoff, and therefore there is no need to specify a utility function. Prices are given as the solution of a two-dimensional non-linear PDE. However, in Hoggard et Al.. (1994) and Dewynne et Al.. (1995), the authors base their model on an assumption of risk neutrality for the trader. Avellaneda and Paras (1994) also assume risk neutrality with respect to hedging residuals, but they consider non-convex payoffs. They study different ways of solving the non-linear PDE when it is ill-posed, by using super-replication. In their setting, it is not worth rehedging sometimes. On the subject of hedging or not, Taleb (1997) considers that in fact market-makers don't pay transaction costs, and those who do have no interest in hedging options. In Henrotte (1994), the author shows the convergence of discrete transaction costs under the assumption the payoff is European, convex, and path-independent. He also studies the behavior of the hedging error at the limit, and the case where portfolio rehedgings are not blindly carried out, but triggered by some price movements.

In Dewynne et Al. (1995), a model is developed which gives the price and hedging strategy in the path-dependent case, but in a particular setting. We note that there are very few approaches which have given closed-form formulae when there are transaction costs, barring the simplest model derived by Leland. On the subject of the convergence of option prices with transaction costs, Kabanov and Mher (1997) have shown that one has to chose a very precise relationship between the frequency of rehedgings and transaction costs so that hedging errors converge to zero if hedging frequency increases indefinitely..

Our target in this chapter is to extend the limit model to the case of a dynamic hedging strategy using an arbitrary set of derivatives, on which transaction costs are very small and hedging frequency is very high. We give a thorough proof of the convergence of the price of the derivatives hedged under these conditions towards a non-linear PDE. This constitutes a generalization of static hedging, which assumes that the replication portfolio, composed of a range of simple derivatives, is not rebalanced over time. Static hedging has been analyzed in Dermann et Al.. (1994) for example.

We will mainly follow Henrotte's approach. We will not study the behavior of hedging errors, but we will show the convergence of transaction costs without neither convex hypotheses nor path-independency hypotheses, in the case of a "blind" hedging strategy (rehedging intervening periodically) and in the case of a more subtle hedging strategy.

The most important aspect of the model is that it will allow us to contemplate hedging not only with the underlying, but with any other simple derivative written on the underlying. Because of the redundancy of the model, it will be possible to still have a perfect replication. This ability to choose to allocate the hedging strategy among different derivatives allows traders to choose the optimal strategy with respect to transaction costs. Their target is to minimize the cumulated transaction costs, knowing that sometimes some products are cheaper than others, that some products better replicate the payoff than others, but their transaction costs structures are different. We will also see how choosing the best products to hedge also allows the existence of the solution to the equation verified by the price.

The paper is organized as follows: in the first section, the model is presented. First, some particular aspects of the standard Black-Scholes model are detailed, so as to be able to model investment into various simple derivatives for hedging purposes. The case of path-dependent payoffs is also examined to give a good basis for the analysis in the case of transaction costs. Then, the transaction costs models are introduced, and prices are expressed as the solutions of non-linear partial differential equations in the path-independent case, and as solution of functional equations in the path-dependent case.

The proof of the convergence result is provided in the second section, and involves a series of thorough applications of Taylor's theorem.

In the third section, We discuss the optimization procedure to follow so as to determine the optimal hedging allocation with respect to transaction costs, and we show the existence of solutions to the pricing equation for European payoffs.

The fourth section concludes the paper, and the fifth section is an appendix, providing the reader with a proof of the convergence of transaction costs in the path-dependent case.

The model

Derivatives redundant hedging

Black-Scholes model assumes, among other things, that it is possible to perfectly replicate the payoff of a derivative product with a self-financing portfolio. If there is only one source of risk, it implies that all the derivatives that can be written on an underlying financial asset are redundant. It makes possible to hedge any derivative with as many other derivatives as one wishes, including the underlying, without changing the price. In Black-Scholes' setting, there is no clear advantage to such a strategy, but it can be of great help if markets are not frictionless, as we will see later on.

The objective in this section is to determine the required amounts of various hedging derivatives to hold in the replication portfolio, so as to obtain a perfect replication, and following, the same price as in Black-Scholes model. We will focus on the simplest approach to model the underlying price, and will use a geometric Brownian Motion.

We start by defining our notations. We introduce a diffusion process to model the underlying price:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t \tag{6.1}$$

where B is a Brownian Motion on the measured space $(\Omega, \mathcal{F}, \mathbb{P})$, and $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration of B. Since there is only one source of noise, the market is complete.

The path-independent case and the partial differential equation approach

We suppose there is a constant risk-free rate r associated to a numeraire process S^0, and we want to hedge and price a derivative product, whose price will be noted $h(t, S_t)$. Indeed, it will be first supposed that the payoff of this derivative product is $\sigma(S_T)$-measurable, with T fixed. We also define a set of plain-vanilla options, including the underlying itself, which can be defined as a call with a null exercise price, indexed by $k \in \Gamma$ where Γ is numerable, and note their prices $C^k(t, S_t)$. These prices follow from the application of Black-Scholes formula. In most cases, the set of derivatives used for hedging would be finite, but perpetual products may require an infinite set. It is possible to generalize all the following arguments to a continuous set of hedging products (but compact), though it does not either respect market practice, nor bring any additional clarity to the analysis.

A hedging strategy based on the underlying financial asset is an adapted process (Δ, Δ^0) such that

$$\begin{aligned} dh(t, S_t) &= \Delta_t dS_t + \Delta_t^0 dS_t^0 \text{ and} \\ h(t, S_t) &= \Delta_t S_t + \Delta_t^0 S_t^0. \end{aligned} \tag{6.2}$$

Δ reflects the number of underlying units which are held in the replication portfolio, and Δ^0 is the amount put in the cash deposit. We also need the following integrability conditions:

$$\begin{aligned} \mathbb{E}\left(\int_0^T \Delta_t^2 dt\right) &< +\infty \text{ and} \\ \mathbb{E}\left(\int_0^T \left(\Delta_t^0\right)^2 dt\right) &< +\infty. \end{aligned}$$

From the definition, and since $dS_t^0 = rS_t^0 dt$, it is clear that

$$\begin{aligned} dh(t, S_t) &= rh(t, S_t)\, dt + \Delta_t dS_t - r\Delta_t S_t dt \\ &= rh(t, S_t)\, dt + \sigma\Delta_t S_t dB_t + \Delta_t \mu S_t dt - r\Delta_t S_t dt. \end{aligned} \tag{6.3}$$

If we consider replicating by means of the various simple options, then we extend the definition of a strategy to a couple (Υ, Δ^0) where Δ^0 is still an adapted process but Υ is an adapted process taking values in $\mathbb{R}^{|\Gamma|}$, such that

$$\begin{aligned} dh(t, S_t) &= \sum_k \Upsilon_t^k dC^k(t, S_t) + \Delta_t^0 r S_t^0 dt \\ h(t, S_t) &= \sum_k \Upsilon_t^k C^k(t, S_t) + \Delta_t^0 S_t^0 \end{aligned} \tag{6.4}$$

which implies

$$dh(t, S_t) = rh(t, S_t)\, dt + \sum_k \Upsilon_t^k dC^k(t, S_t) - r\sum_k \Upsilon_t^k C^k(t, S_t)\, dt \tag{6.5}$$

Implicitly, it is assumed that the simple options are only used in replacement of the financial underlying part of the replication strategy, and the balance between cash investment and risky investment remains the same. The investor has an absolute freedom as for the choice of the allocation amongst the different hedging products. In any case the price remains the same, as options are redundant. Using Itô's theorem, we can write that

$$\begin{aligned} dC^k(t, S_t) &= C_t^k(t, S_t)\, dt + C_S^k(t, S_t)\, \mu S_t dt + C_S^k(t, S_t)\, \sigma S_t dB_t \\ &\quad + \frac{1}{2} S_t^2 \sigma^2 C_{SS}^k(t, S_t)\, dt. \end{aligned} \tag{6.6}$$

Then, by inverting the sum and the stochastic integral[2], we have

$$\begin{aligned} \sum_k \Upsilon_t^k dC^k(t, S_t) &= \sum_k \Upsilon_t^k C_t^k(t, S_t)\, dt + \sum_k \Upsilon_t^k C_S^k(t, S_t)\, \mu S_t dt \\ &\quad + \sum_k \Upsilon_t^k C_S^k(t, S_t)\, \sigma S_t dB_t \\ &\quad + \frac{1}{2}\sum_k \Upsilon_t^k S_t^2 \sigma^2 C_{SS}^k(t, S_t)\, dt. \end{aligned} \tag{6.7}$$

[2] Using a special Fubini theorem, cf Protter (1994), p. 160. We use the fact that the numerable sum is a particular case of integration with a positive finite measure (the Dirac mass).

Again, thanks to Itô's theorem, we know that

$$\begin{aligned} dh(t,S_t) &= h_t(t,S_t)\,dt + h_S(t,S_t)\,dS_t + \frac{1}{2}h_{SS}(t,S_t)\,d[S]_t \\ &= h_t(t,S_t)\,dt + h_S(t,S_t)\,\mu S_t dt + h_S(t,S_t)\,\sigma S_t dB_t \\ &\quad + \frac{1}{2}S_t^2\sigma^2 h_{SS}(t,S_t)\,dt. \end{aligned} \tag{6.8}$$

Using Doob-Meyer's identity theorem (cf Protter (1994), p. 94), we obtain from the above equations that the following identity must be verified:

$$h_S(t,S_t) = \sum_k \Upsilon_t^k C_S^k(t,S_t)\,. \tag{6.9}$$

If $C_S^k(t,S_t) \neq 0$, we can choose $\Upsilon_t^k = \eta_t^k \frac{h_S(t,S_t)}{C_S^k(t,S_t)}$, which is consistent with the identity, if $\sum_k \eta_t^k = 1$. To be sure that the delta of the plain-vanilla options is not zero, we can suppose that they are sold just before their maturity[3]. It is therefore possible to hedge a long maturity option with shorter ones, as far as these options are phased out of the hedging portfolio -strictly- before maturity. The problem posed by hedging with shorter-lived options is that the amount received at maturity is "discrete", in the sense that it is not infinitesimal. The set of hedging options being discrete, it comes down to shifting from the position taken in the short-lived option to the other ones, which are still alive, exactly when the payoff has been paid. This puts a special constraint on η, the "short-lived component" of which must then be set at zero at maturity date. The problem we face is also that if, as we will do it later, we consider that transactions costs exist but are very small, then a finite number of discrete transactions, with non-infinitesimal amounts, will not cost anything, whereas the same amount traded little by little, as an infinite number of infinitesimal transactions, will cost some money, which is exactly the effect we want to study. In fact, it boils down to setting an upper limit in the speed at which transactions can be carried out.

It appears clearly that if some of the hedging products are shorter-lived than the derivative hedged by the trader, then there will be at most a finite number of shifts in the position held by the trader, and we do not mind having a finite number of finite-sized transactions. Indeed, we will see later that in that case, and under our hypotheses, the induced costs will be negligeable.

We also add the constraint that $\eta \geq 0$. As we are in the case of a non-path-dependant derivative, we can restrict ourselves to the strategies η such that $\eta_t^k = \eta^k(t,S_t)$.

As we want to avoid trading big amounts at the same time, and prefer only trading infinitesimal amounts, as it is the case in the standard Black-Scholes model, then it is important that the strategy $\eta^k(t,S_t)$ should be differentiable with respect to t and S, as we will see when we take into account the effects of transaction costs.

We have therefore clearly defined a hedging strategy for non-path-dependant assets using various simple derivative products as a simple set of weights.

The path-dependant case, with the Risk-Neutral probability approach

We are now trying to write a hedging strategy in the case of a path-dependant product. In the classical model, the price corresponds to the expectation of the

[3] That is, we can fix a small positive number ρ such that all options are sold ρ units of time before their maturity.

payoff under a particular probability, and we expect to obtain the same result in our new setting.

First of all, we need to detail a few points about the original model. Unlike the preceding model in which the use of Itô's theorem has allowed us to identify the replication strategy with the derivative of the price with respect to the underlying, the risk-neutral probability approach does not give this result in itself. Indeed, the identification of a probability measure such that the present values of prices are martingales along with the martingale representation theorem only allow us to write the price of an $\mathcal{F}_T$-measurable random variable and not the replication strategy. But it is widely used that this strategy is still the derivative of the price with respect to the underlying. We are going to clarify this point and see which hypotheses are sufficient to have this result.

We consider the simplest case, that of a $\mathcal{F}_T$-measurable, square integrable, payoff X. The self-financing portfolio (as we do not introduce transaction costs yet) has the value V.

Remark 3 *The implicit assumption made is the following: V and the final payoff X depend on T, S_T, and H_T where H is a finite variation adapted process*[4] *and (S, H) is Markovian.*

We still write the same accounting equation for the portfolio:

$$\begin{aligned} dV &= \Delta dS + rVdt - r\Delta Sdt \\ V(T) &= X \end{aligned} \tag{6.10}$$

but now, the application of Itô's theorem gives, since H is a finite variation process

$$dV = V_S dS + V_t dt + \frac{1}{2} V_{SS} d[S] + V_H dH. \tag{6.11}$$

And by identification we have the result, that is $\Delta = V_S$. Of course, we cannot get, properly speaking, a real PDE as before, since it happens to be rather a functional equation, due to the intervention of H.

Now we can return to the standard approach for pricing. We know, thanks to Girsanov's theorem and the martingale representation theorem, that there exists a unique probability measure $\mathbb{Q}$ such that under $\mathbb{Q}$,

- $\left(B_t + \frac{\mu - r}{\sigma} t = Z_t\right)_{t \geq 0}$ is a Brownian Motion
- For any price V associated to a replication strategy Δ, $\frac{V}{S^0}$ is a $\mathbb{Q}$-local martingale
- $V(t, S_t, H_t) = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)} X \middle| \mathcal{F}_t \right]$

This approach does not give directly the replication strategy, but we have exposed a way of writing it.

Now, let us return to the more general case of the extended replication strategy. We know that the price remains the same if we use redundant assets for hedging purposes, and still corresponds to the expectation under the risk-neutral

[4] That is, a process which equals the difference of two increasing processes. It can be checked that for "usual" exotic products, the part of the price which is a functional of the path of the underlying constitutes a finite variation process.

probability, but we have to determine the amounts invested in the different assets. We can write the evolution of the value of a strategy as

$$\begin{aligned} dh(\Delta)_t &= rh(\Delta)_t\,dt + \sum_k \Upsilon_t^k dC^k(t,S_t) - r\sum_k \Upsilon_t^k C^k(t,S_t)\,dt \\ &= rh(\Delta)_t\,dt + \Delta_t dS_t - rS_t\Delta_t \\ h(\Delta)_T &= X \end{aligned} \tag{6.12}$$

now associated to an $\mathcal{F}_T$-measurable payoff X.

An example of a financial product depending on S and H would be a lookback option with maturity T. Its payoff would be $(S_T - \inf_{0\leq s\leq T} S_s)^+$, or $(S_T - H_T)^+$ with $H_t = \inf_{0\leq s\leq t} S_s$ which is a finite variation process. When there are no transaction costs, the price at time $t < T$ of the lookback option is a known smooth function of S_t and H_t. Another example is the case of an Asian option (or average option). The payoff in this case does not depend on S, but on the process $H_t = \int_0^t S_s dt$, which has finite variations. Again another example would be an option of the maximum: an interesting case for investors who leverage on volatility is $H_t = \sup_{0\leq s\leq t} S_s$ -$\inf_{0\leq s\leq t} S_s$.

Let us now have a closer look at the replication strategy, so as to see what is changed if we are allowed to use other financial assets than the underlying. We make the same assumptions as earlier, that is the price h is supposed to depend on a finite variation process representing the path-dependency of the payoff. We can write the two versions of dh as earlier, and check immediately by identification that we have to chose

$$\begin{aligned} \Upsilon_t(dk) &= \eta_t^k \frac{h_S(t,S_t,H_t)}{C_S^k(t,S_t)} \text{ and therefore} \\ h_S(t,S_t) &= \sum_k \Upsilon_t^k C_S^k(t,S_t) \end{aligned} \tag{6.13}$$

for the extended replication strategy. We can consider that $\eta_t = \eta(t,S_t,H_t)$ to take into account the effects of path-dependency. It is also natural to add the constraints we have presented earlier.

Leland's hedging strategy

In this section we use the preceding setting in a particular case of transaction costs derived from Leland's model. We consider the case when transaction costs are proportional and very small, and hedging frequency is very high. We will see that at the limit, when transaction costs are paid at a continuous rate it is possible to build a portfolio that perfectly replicates the derivative payoff.

Henrotte (1994) and Ahn et Al.. (1996) study the limit behavior of the variance between the payoff of an option and the value of the replication portfolio. It has appeared difficult in the generalized setting we took, to extend the results derived in the above mentioned papers. Henrotte's assumption of a convex payoff also fits poorly with the reality of a traded portfolio. Indeed, it is highly probable that a trader will hold puts as well as calls of different strike prices, written on the same maturity. The problem is that as soon as transaction costs intervene, the pricing operator associating a price to a random variable is no more linear, due to the absence of self-financing. Then it is not possible to make a breakdown of the portfolio into convex and concave payoffs, which in return makes it almost impossible to use the results assuming convex payoffs in practice. This non-linearity also implies the following natural remark: the minimum (break-even) price at which a trader can sell a derivative depends on the composition of his portfolio at this time. There are indeed obvious economies for the transaction costs to pay if a product to sell perfectly fits into the "holes" left by the ones already in the portfolio.

We consider that transaction costs as well as the constant time between two portfolio rehedgings depend on a small parameter ε, such that both converge towards 0 with ε, with their ratio remaining constant. Then, at the limit, hedging is perfect, and transaction costs are paid in a continuous mode. We intend to use such a limit model, considering that transaction costs are small enough so that the limit model is a good proxy for the non-limit one, and the frequency of rehedging is high enough so that the hedging error may be neglected.

We write D_t^ε the value of the portfolio in the non-limit case, that the trader constitutes to attempt to hedge the derivative. It is not a perfect hedge since rebalancings intervene only at discrete times. We will show that the value of the portfolio in the non-limit case converges in some suitable sense to h, the value in the limit case.

First, we write Δt the delay between two consecutive rehedgings of the portfolio, for a given maturity T. We define the proportional transaction costs to be $\varepsilon\delta^k\left(t, C_t^k\right)$, depending on the product k being considered. We write $\tau^k\left(t, C_t^k\right) = \frac{\varepsilon\delta^k\left(t, C_t^k\right)}{\sqrt{\Delta t}}$, which is assumed to remain constant when ε and Δt both tend to zero. This gives an implicit definition of τ^k.

We suppose that in the non-limit case, the trader uses the limit case optimal hedging strategy. If the parameters are small enough, then the hedging error is negligeable, and this hedging strategy is relevant. The trader is in fact supposed to be risk-neutral with respect to the residual hedging error in the non-limit case. At the limit, this assumption is not needed anymore as replication is perfect.

We can write

$$\begin{aligned}
D_T^\varepsilon = {} & h\left(0, S_0\right) \\
& + \sum_{t_j \leq T} \sum_k \eta^k\left(t_j, S_j\right) \frac{h_S\left(t_j, S_j\right)}{C_S^k\left(t_j, S_j\right)} \left(C^k\left(t_{j+1}, S_{j+1}\right) - C^k\left(t_j, S_j\right)\right) \\
& - \sum_{t_j \leq T} \sum_k \left| \eta^k\left(t_{j+1}, S_{j+1}\right) \frac{h_S\left(t_{j+1}, S_{j+1}\right)}{C_S^k\left(t_{j+1}, S_{j+1}\right)} - \eta^k\left(t_j, S_j\right) \frac{h_S\left(t_j, S_j\right)}{C_S^k\left(t_j, S_j\right)} \right| \\
& \times \sqrt{\Delta t} \tau^k\left(t_{j+1}, C_{j+1}^k\right) C^k\left(t_{j+1}, S_{j+1}\right) \\
& + \sum_{t_j \leq T} \left(h\left(t_j, S_j\right) - S_j h_S\left(t_j, S_j\right)\right) \frac{\left(S_{j+1}^0 - S_j^0\right)}{S_j^0}.
\end{aligned} \tag{6.14}$$

The first sum is the value of the part of the portfolio which tries to track the payoff of the derivative, using various other options. The second sum is the cumulated amount of the transactions costs induced by the hedging strategy, and the third sum is the amount invested in the cash deposit, that we have written as the whole value minus the amount invested in risky assets. The transaction costs are paid on all the amounts transferred from risky assets to cash and reciprocally; and reweighing the position from one risky asset to the other has to be done via the cash deposit. It appears clearly in 6.14 that all these possible movements are accounted for. We now state a first result, based on the convergence Theorem 23 shown in the next section.

Theorem 19 *Assuming that*

- *h and η belong to $C^{3,3}$, with bounded derivatives*
- *τ^k, for each k, belongs to $C^{2,2}$*

then the value of the derivative when transaction costs and the delay between portfolio rehedgings tend together to zero verifies the following partial differential equation:

$$r\left(Sh_S - h\right) + h_t + \frac{\sigma^2}{2}S^2 h_{SS} = \sqrt{\frac{2}{\pi}}\sigma \sum_k \tau^k C^k S \left| \eta_S^k \frac{h_S}{C_S^k} - \eta^k \frac{h_{SS}C_S^k - h_S C_{SS}^k}{\left(C_S^k\right)^2} \right|.$$

Proof. We start by applying Theorem 23 on p. 100. As the assumptions are verified for this theorem, we know that for each k

$$\begin{aligned} &\sum_{t_j \le T} \left| \eta^k\left(t_{j+1}, S_{j+1}\right) \frac{h_S\left(t_{j+1}, S_{j+1}\right)}{C_S^k\left(t_{j+1}, S_{j+1}\right)} - \eta^k\left(t_j, S_j\right) \frac{h_S\left(t_j, S_j\right)}{C_S^k\left(t_j, S_j\right)} \right| \\ &\times \sqrt{\Delta t}\tau^k\left(t_{j+1}, C_{j+1}^k\right) C^k\left(t_{j+1}, S_{j+1}\right) \end{aligned} \tag{6.15}$$

converges in L^1 towards

$$\sqrt{\frac{2}{\pi}}\sigma \int_0^T \tau^k C^k S \left| \eta_S^k \frac{h_S}{C_S^k} - \eta^k \frac{h_{SS}C_S^k - h_S C_{SS}^k}{\left(C_S^k\right)^2} \right| ds. \tag{6.16}$$

On the other hand, standard stochastic integration theory states that for each k

$$\sum_{t_j \le T} \eta^k\left(t_j, S_j\right) \frac{h_S\left(t_j, S_j\right)}{C_S^k\left(t_j, S_j\right)} \left(C^k\left(t_{j+1}, S_{j+1}\right) - C^k\left(t_j, S_j\right)\right) \tag{6.17}$$

converges in L^2 to $\int_0^T \eta^k \frac{h_S}{C_S^k} dC^k$, which by definition equals $\int_0^T h_S dS$.

It is also clear that the sum

$$\sum_{t_j \le T} \left(h\left(t_j, S_j\right) - S_j h_S\left(t_j, S_j\right)\right)\left(S_{j+1}^0 - S_j^0\right) \tag{6.18}$$

converges in the same way to $r\int_0^T \left(h - Sh_S\right) ds$.

Summing over k does not change the result, and we obtain that the value of the replication portfolio D_T^ε converges in L^1 to

$$\begin{aligned} h\left(T, S_T\right) &= \sum_k \int_0^T \eta^k \frac{h_S}{C_S^k} dC^k + r\int \left(h - Sh_S\right) ds \\ &\quad + \sqrt{\frac{2}{\pi}}\sigma \sum_k \int_0^T \tau^k C^k S \left| \eta_S^k \frac{h_S}{C_S^k} - \eta^k \frac{h_{SS}C_S^k - h_S C_{SS}^k}{\left(C_S^k\right)^2} \right| ds \\ &= \int_0^T h_S dS + r\int_0^T \left(h - Sh_S\right) ds \\ &\quad + \sqrt{\frac{2}{\pi}}\sigma \sum_k \int_0^T \tau^k C^k S \left| \eta_S^k \frac{h_S}{C_S^k} - \eta^k \frac{h_{SS}C_S^k - h_S C_{SS}^k}{\left(C_S^k\right)^2} \right| ds. \end{aligned} \tag{6.19}$$

But, thanks to Itô's theorem, we know that

$$h(T, S_T) = h(0, S_0) + \int_0^T h_S dS + \int_0^T h_t dt + \frac{\sigma^2}{2} \int_0^T h_{SS} S^2 ds. \tag{6.20}$$

Finally, with the help of the Meyer's identification theorem (cf. Protter, 1994), we can write the expected PDE ■

The risk-related hedging strategy

Following Henrotte (1994) we study the case where the portfolio is rehedged at certain times, depending on the evolution of the underlying's price. In his paper, Henrotte deals with a European, path-independent, and convex payoff option. As we wished to price non-convex portfolios, it has been necessary to reduce the set of possible triggering events for the reweighing to take place.

We now consider the following hedging strategy: as soon as some particular measure of a deviation from the ideal quantity of underlying holding deviates for more than a given percentage, the holding of this underlying are changed to the ideal ones. We also suppose that the right quantity of holding is given by the delta, that is, the derivative of the price with respect to the underlying's price. Indeed, we consider that the trader wants to minimize his local risk, as in Hoggard et Al.. (1994), as opposed to maximizing a utility function on terminal wealth. It is clear that if the percentage in question is non-zero, then hedging is not perfect and there is a replication error.

We consider a portfolio whose payoff depends on the underlying in a European way, without depending on the path followed by the underlying. We write the price of the portfolio at the limit $h(t, S_t)$. h is assumed to be smooth enough so that we can define its successive derivatives. Again, we write D_t^ε the value of the replication portfolio, in the non-limit case. We will show that the value of the portfolio in the non-limit case converges to h.

We define, as in the preceding sub-section, the proportional transaction costs to be $\varepsilon\delta^k\left(t, C_t^k\right)$, a^k the percentage deviation at which the portfolio is hedged, and b^k such that $1 - e^{-a^k} = e^{b^k} - 1$. We write $\tau^k(t, S_t) = \frac{\varepsilon\delta^k\left(t, C_t^k\right)}{1-e^{-a^k}}$ which remains constant when ε and a both tend to zero. We also note the hedging instants t_j^k, which are defined for a positive real number ξ as $t_0^k = 0$ and

$$\begin{aligned}
t_{j+1}^k \quad = \quad & T \wedge \inf \Bigg\{ t \geq t_j^k : \eta^k(t, S_t) \frac{h_S(t, S_t)}{C_S^k(t, S_t)} = \\
& \mathbb{I}_{\left|\eta^k(t_j, S_j) \frac{h_S(t_j, S_j)}{C_S^k(t_j, S_j)}\right| > \xi} \eta^k(t_j, S_j) \frac{h_S(t_j, S_j)}{C_S^k(t_j, S_j)} e^{-a^k} \\
& + \mathbb{I}_{\left|\eta^k(t_j, S_j) \frac{h_S(t_j, S_j)}{C_S^k(t_j, S_j)}\right| \leq \xi} \left(\eta^k(t_j, S_j) \frac{h_S(t_j, S_j)}{C_S^k(t_j, S_j)} + \xi \left(1 - e^{-a^k}\right) \right) \\
& \text{or } \mathbb{I}_{\left|\eta^k(t_j, S_j) \frac{h_S(t_j, S_j)}{C_S^k(t_j, S_j)}\right| > \xi} \eta^k(t_j, S_j) \frac{h_S(t_j, S_j)}{C_S^k(t_j, S_j)} e^{b^k} \\
& + \mathbb{I}_{\left|\eta^k(t_j, S_j) \frac{h_S(t_j, S_j)}{C_S^k(t_j, S_j)}\right| \leq \xi} \left(\eta^k(t_j, S_j) \frac{h_S(t_j, S_j)}{C_S^k(t_j, S_j)} + \xi \left(e^{b^k} - 1\right) \right) \Bigg\}
\end{aligned} \tag{6.21}$$

This definition implies that the portfolio is rehedged with respect to any of the underlyings derivatives under consideration as soon as the optimal quantity which should be held has moved by a certain amount. Be this variation due to a necessary rehedging or to a change in the decision parameter given by the measure η, the part of the portfolio in question is readjusted. We avoid problems when the holdings tend to zero, in which case it is not possible to refer to a percentage variation of the delta. We consider that as soon as the delta is sufficiently close to zero, the variation becomes an absolute value. Note that if we are in a situation such that for all t

$$\eta^k (t, S_t) \frac{h_S (t, S_t)}{C_S^k (t, S_t)} > 0 \tag{6.22}$$

then this problem does not appear.

We consider that in the non-limit case, the derivative portfolio is sold at the price it would have in the limit-case, that is h. Therefore we can write

$$\begin{aligned} D_T^{\varepsilon} &= h(0, S_0) \\ &+ \sum_{t_j \leq T} \sum_k \eta^k (t_j, S_j) \frac{h_S (t_j, S_j)}{C_S^k (t_j, S_j)} \Big(C^k (t_{j+1}, S_{j+1}) - C^k (t_j, S_j) \Big) \\ &- \sum_{t_j \leq T} \sum_k \left| \eta^k (t_{j+1}, S_{j+1}) \frac{h_S (t_{j+1}, S_{j+1})}{C_S^k (t_{j+1}, S_{j+1})} - \eta^k (t_j, S_j) \frac{h_S (t_j, S_j)}{C_S^k (t_j, S_j)} \right| \\ &\times \left(1 - e^{-a^k}\right) \tau^k \left(t_{j+1}, C_{j+1}^k\right) C^k (t_{j+1}, S_{j+1}) \\ &+ \sum_{t_j \leq T} (h(t_j, S_j) - S_j h_S (t_j, S_j)) \frac{\left(S_{j+1}^0 - S_j^0\right)}{S_j^0}. \end{aligned} \tag{6.23}$$

The following result relies on Theorem 24 shown in the next section.

Theorem 20 *Assuming that*

- *h and η belongs to $C^{3,3}$, with bounded derivatives*
- *τ^k, for each k, belong to $C^{2,2}$*

then the value of the derivative when transaction costs and the rehedging spread tend together to zero verifies the following partial differential equation:

$$r(Sh_S - h) + h_t + \frac{\sigma^2}{2} S^2 h_{SS} = \sigma^2 \sum_k \frac{S^2 \tau^k C^k \left(\eta_S^k \frac{h_S}{C_S^k} + \eta^k \frac{h_{SS} C_S^k - h_S C_{SS}^k}{\left(C_S^k\right)^2} \right)^2}{\mathbb{I}_{\left|\eta^k \frac{h_S}{C_S^k}\right| > \xi} \left| \eta^k \frac{h_S}{C_S^k} \right| + \xi \mathbb{I}_{\left|\eta^k \frac{h_S}{C_S^k}\right| \leq \xi}}.$$

Proof. By definition, we have

$$\left(1 - e^{-a^k}\right) = \frac{\left| \eta^k (t_{j+1}, S_{j+1}) \frac{h_S (t_{j+1}, S_{j+1})}{C_S^k (t_{j+1}, S_{j+1})} - \eta^k (t_j, S_j) \frac{h_S (t_j, S_j)}{C_S^k (t_j, S_j)} \right|}{\mathbb{I}_{\left|\eta^k (t_j, S_j) \frac{h_S (t_j, S_j)}{C_S^k (t_j, S_j)}\right| > \xi} \left| \eta^k (t_j, S_j) \frac{h_S (t_j, S_j)}{C_S^k (t_j, S_j)} \right| + \xi \mathbb{I}_{\left|\eta^k (t_j, S_j) \frac{h_S (t_j, S_j)}{C_S^k (t_j, S_j)}\right| \leq \xi}}. \tag{6.24}$$

Therefore we can write

$$\begin{aligned}
D_T^\varepsilon &= h(0,S_0) \\
&+\sum_{t_j\le T}\sum_k \eta^k(t_j,S_j)\frac{h_S(t_j,S_j)}{C_S^k(t_j,S_j)}\left(C^k(t_{j+1},S_{j+1})-C^k(t_j,S_j)\right) \\
&-\sum_{t_j\le T}\sum_k\left(\eta^k(t_{j+1},S_{j+1})\frac{h_S(t_{j+1},S_{j+1})}{C_S^k(t_{j+1},S_{j+1})}-\eta^k(t_j,S_j)\frac{h_S(t_j,S_j)}{C_S^k(t_j,S_j)}\right)^2 \\
&\times\frac{\tau^k\left(t_{j+1},C_{j+1}^k\right)C^k(t_{j+1},S_{j+1})}{\mathbb{I}_{\left|\eta^k(t_j,S_j)\frac{h_S(t_j,S_j)}{C_S^k(t_j,S_j)}\right|>\xi}\left|\eta^k(t_j,S_j)\frac{h_S(t_j,S_j)}{C_S^k(t_j,S_j)}\right|+\xi\mathbb{I}_{\left|\eta^k(t_j,S_j)\frac{h_S(t_j,S_j)}{C_S^k(t_j,S_j)}\right|\le\xi}} \\
&+\sum_{t_j\le T}\left(h(t_j,S_j)-S_jh_S(t_j,S_j)\right)\frac{\left(S_{j+1}^0-S_j^0\right)}{S_j^0}. \qquad (6.25)
\end{aligned}$$

Using Theorem 24 (p. 106 adapted from Henrotte's paper, we get that D_T^ε converges in probability over compacts to

$$\begin{aligned}
h(T,S_T) &= \sum_k\int_0^T\eta^k\frac{h_S}{C_S^k}dC^k+r\int_0^T(h-Sh_S)\,ds \\
&+\sum_k\int_0^T\frac{\tau^kC^k}{\mathbb{I}_{\left|\eta^k\frac{h_S}{C_S^k}\right|>\xi}\left|\eta^k\frac{h_S}{C_S^k}\right|+\xi\mathbb{I}_{\left|\eta^k\frac{h_S}{C_S^k}\right|\le\xi}}d\left[\eta^k\frac{h_S}{C_S^k}\right]_s \\
&= \int_0^T h_S dS+r\int_0^T(h-Sh_S)\,ds \\
&+\sigma^2\sum_k\int_0^T\frac{S^2\tau^kC^k\left(\eta_S\frac{h_S}{C_S^k}+\eta^k\frac{h_{SS}C_S^k-h_SC_{SS}^k}{\left(C_S^k\right)^2}\right)^2}{\mathbb{I}_{\left|\eta^k\frac{h_S}{C_S^k}\right|>\xi}\left|\eta^k\frac{h_S}{C_S^k}\right|+\xi\mathbb{I}_{\left|\eta^k\frac{h_S}{C_S^k}\right|\le\xi}}ds. \qquad (6.26)
\end{aligned}$$

The sum accounting for the amounts invested in the cash deposit trivially converges to the same limit as in the preceding theorem.

But, once again, thanks to Itô's theorem, we know that

$$h(T,S_T)=h(0,S_0)+\int_0^T h_SdS+\int_0^T h_tdt+\frac{\sigma^2}{2}\int_0^T h_{SS}S^2ds \qquad (6.27)$$

And by identification we obtain the result. ∎

For technical reasons in the proof of the theorem we demanded that the allocation η^k be differentiable with respect to time. It is obvious that the differentiability with respect to the underlying's price is necessary, since this derivative appears in the final formula, but one can wonder why the time differential does not appear. An intuitive explanation is the following: in our setting, we are trading an infinitely big number of infinitely small transactions, and finally get a finite price. If there were two domains in the (t,S_t) plane such that in one the replication portfolio is fully weighted in some option, and in the other one it is fully weighted in some other option (in which case the measure would not be differentiable with respect to the underlying), then, because of the local time spent by the process at the border between the two domains, there would be an infinite number of finite sized transactions (reweighing the whole portfolio at each time). On the other hand, if we consider a strategy which is not time-differentiable on a finite number of instants, we will pay a finite number of finite sized transactions, which is more bearable. At last, since transaction costs are supposed to converge to zero, the amount paid a finite number of times tend to zero.

The continuity hypotheses we made on the price function, in both cases (blind or not hedging strategy), is in fact fully compatible with non-differentiable payoffs a priori only in the case when there are no transaction costs. Indeed, they are relevant to the price of the securities on an open interval, which can accommodate non-differentiable shapes at the extremity of the interval, as the PDE they verify can be solved with non smooth boundary conditions. When there are transaction costs, non-convex or non-smooth payoffs pose a problem: they cannot be replicated correctly, as exposed in Avellaneda and Paras (1994) or Dewynne et Al. (1995) due to ill-posedness, and require special hedging strategies.

In the third section, we study under which conditions there will be solutions to the pricing equations of theorems 19 and 20.

The path-dependent case with transaction costs

We focus now on how to be able to specify the replication strategy in the same manner as we did before. We follow and extend the approach developed in the path-independent case.

We start by defining, with the intuition of the above results, the notion of transaction costs associated to the replication of a payoff. We also make the hypothesis that the price of the derivative is a function $p(t, S_t, H_t)$ where H is a finite variation process.

Definition 21 *A transaction cost functional is a positive functional K associated to a replication strategy.*

For example, it can be associated to the gamma of the option being hedged. In Dewynne et Al. (1995), the simplest case mentioned is that of transaction costs totaling $cS^2 |p_{SS}|$ for some constant c. The transaction cost functional in that case is therefore $Kf = c \left| \frac{\partial f}{\partial x} \right| x^2$.

Theorem 22 *The price p of an exotic payoff X, under a transaction cost functional K, must verify the following functional equation*

$$\begin{aligned} p(t, S_t, H_t) &= \mathbb{E}_{\mathbb{Q}} \left[e^{-r(T-t)} X \middle| \mathcal{F}_t \right] \\ &\quad + \mathbb{E}_{\mathbb{Q}} \left[\int_t^T ds e^{-rs} K(p_S)(s, S_s, H_s) \middle| \mathcal{F}_t \right]. \end{aligned}$$

Proof. We write the following equation for the value of the portfolio, associated to a replication strategy Δ

$$\begin{aligned} dp &= rpdt + \Delta dS - r\Delta S dt - K(\Delta)\, dt \\ X &= p(T, S_T, H_T). \end{aligned} \tag{6.28}$$

Writing Itô's theorem for p and using the identification theorem gives that necessarily we still have $\Delta = p_S$. Therefore, we have identified the replication strategy. We now have to write the price.

Switching to the unique risk-neutral probability, we can write

$$dp = rpdt + \sigma Sp_S dZ - K(p_S)\,dt \tag{6.29}$$

and therefore, if we write the same equation for the present value, we obtain

$$d\left(\frac{p}{S^0}\right) = \sigma Sp_S dZ - \frac{K(p_S)}{S^0}dt. \tag{6.30}$$

This last equation can also be written

$$\begin{aligned} & e^{-rT}p(T,S_T,H_T) - e^{-rt}p(t,S_t,H_t) \\ = & \int_t^T \sigma S_s p_S(s,S_s,H_s)\,dZ_s - \int_t^T e^{-r(s-t)}K(p_S)(s,S_s,H_s)\,ds \end{aligned} \tag{6.31}$$

and by taking the expectation under the risk-neutral probability conditional on $\mathcal{F}_t$, we get

$$\begin{aligned} p(t,S_t,H_t) &= \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}X\middle|\mathcal{F}_t\right] \\ &\quad +\mathbb{E}_{\mathbb{Q}}\left[\int_t^T dse^{-rs}K(p_S)(s,S_s,H_s)\middle|\mathcal{F}_t\right] \end{aligned} \tag{6.32}$$

which is the expected result. ■

This result is, of course, applicable in the case of path-independent payoffs, that is if H is zero. As we will see later, it can also be shown thanks to Feynman-Kac's theorem.

Now, we have to specify the transaction cost functional K. We will assume, by analogy with the results we have obtained in the path-independent case, that K has the same shape in the path-dependent case. The demonstration that the cumulated transaction costs in the non-limit case, when the price depends on a path functional H, converge to the same expression as in the path-independent case is based on Theorem 27 on p. 114, which is an extension of Theorem 23. But this extension uses the hypothesis that the process H, representing the path-dependency, is continuous. An extension to the case with bounded jumps seems to be difficult. For the extension of Theorem 24, we rely on a very intuitive argument (as there does not seem to be a need to rewrite the proof).

Endowed with these generalized convergence results, we can prove that, even if there is a dependency on the past evolution of S, the cumulated transaction costs converge to a similar expression as in the path-independent case.

Due to the form of the functional pricing equation obtained if the payoff is path-dependent, showing the existence and unicity of solutions is an open question, which for now has to be dealt with on a case-by-case basis.

Proof of the convergence results

The following result has been used in the proof of Theorem 19. It allowed us to express the limit value of total transaction costs in a handy way, in the case where the derivative price depends only on the underlying's price, and not its trajectory. The theorem and the proof in the case of a path-dependent derivative are given in the appendix.

Theorem 23 *Let $(t_i^n, 0 \le i \le n)_{n\in\mathbb{N}}$ a series of uniform partitions of a positive interval I, F and G be two $C^{2,2}$ functions with bounded derivatives, F being positive, and S a continuous diffusion such that*

- $\lim_{n\to\infty}\sup_{i\leq n}\left(t_{i+1}^n - t_i^n\right) = 0$
- $dS_t = S_t\mu dt + S_t\sigma dB_t$ *where* B *is a Brownian Motion*
- $\mathbb{E}\int_I F(s,S_s)\,ds < \infty$ *and* $\mathbb{E}\int_I S_s^2 F^2(s,S_s)\left|G_S(s,S_s)\right|^2 ds < \infty$

Then we have the following convergence result:

$$\sum_{i=0}^{n}\sqrt{t_{i+1}^n - t_i^n}F\left(t_{i+1}^n, S_{t_{i+1}^n}\right)\left|G\left(t_{i+1}^n, S_{t_{i+1}^n}\right) - G\left(t_i^n, S_{t_i^n}\right)\right|$$

converges in $L^1(\Omega,\mathbb{P})$ *to*

$$\sqrt{\frac{2}{\pi}}\int_I dsF(s,S_s)\,\sigma S_s\left|G_S(s,S_s)\right|.$$

Proof. The demonstration is based on Taylor's theorems. We are interested in the following quantity, where we have naturally simplified the notations:

$$\mathbb{E}\left[\left|\sum_{t_i,i\leq n}\sqrt{t_{i+1}-t_i}F_{i+1}\left|G_{i+1}-G_i\right| - \sigma\sqrt{\frac{2}{\pi}}\int_I dsF(s,S_s)\,S_s\left|G_S(s,S_s)\right|\right|\right]. \tag{6.33}$$

We do not bound on the upper side this amount by extracting the sum from the absolute value, as it would not be possible, at the end, to show its convergence towards 0. Instead, we study the random variable

$$\left|\sum_{t_i,i\leq n}\sqrt{t_{i+1}-t_i}F_{i+1}\left|G_{i+1}-G_i\right| - \sigma\sqrt{\frac{2}{\pi}}\int_I dsF(s,S_s)\,S_s\left|G_S(s,S_s)\right|\right|. \tag{6.34}$$

As a first step, we notice that for a continuous function f,

$$\left|\int_{t_i}^{t_{i+1}} ds\left(f(s,S_s) - f(t_i,S_i)\right)\right| \leq (t_{i+1}-t_i)\sup_{t_i\leq s\leq t_{i+1}}\left|\left(f(s,S_s) - f(t_i,S_i)\right)\right| \text{ a.s.} \tag{6.35}$$

Hence, we obtain that almost surely,

$$\begin{aligned}&\left|\int_{t_i}^{t_{i+1}} dsF(s,S_s)\,S_s\left|G_S(s,S_s)\right| - (t_{i+1}-t_i)\,F_iS_i\left|G_S(t_i,S_i)\right|\right|\\ \leq\ & (t_{i+1}-t_i)\sup_{t_i\leq s\leq t_{i+1}}\left|F(s,S_s)\,S_s\left|G_S(s,S_s)\right| - F_iS_i\left|G_S(t_i,S_i)\right|\right|.\end{aligned} \tag{6.36}$$

Using Taylor's theorem, we also have

$$\begin{aligned}&\left|F\left(t_{i+1},S_{t_{i+1}}\right) - F(t_i,S_{t_i}) - (t_{i+1}-t_i)\,F_t(t_i,S_{t_i}) - (S_{i+1}-S_i)\,F_S(t_i,S_{t_i})\right|\\ \leq\ & \frac{\left((t_{i+1}-t_i)^2 + (S_{i+1}-S_i)^2\right)}{2}\sup_{z,t_i\leq s\leq t_{i+1}}\left(\left|F_{tt}(s,z)\right| + \left|F_{SS}(s,z)\right| + 2\left|F_{tS}(s,z)\right|\right)\\ \leq\ & M\left((t_{i+1}-t_i)^2 + (S_{i+1}-S_i)^2\right)\end{aligned} \tag{6.37}$$

because of the hypothesis that the derivatives are bounded. From this result we deduce that

$$\left|F\left(t_{i+1},S_{t_{i+1}}\right) - F(t_i,S_{t_i})\right| \leq M\left((t_{i+1}-t_i) + \left|S_{i+1}-S_i\right|\right) \tag{6.38}$$

as well as

$$\left|G\left(t_{i+1},S_{t_{i+1}}\right)-G\left(t_i,S_{t_i}\right)-\Delta S_i G_S\left(t_i,S_i\right)\right|\leq M\left(\Delta t+\Delta S_i^2\right) \tag{6.39}$$

and

$$\left|G\left(t_{i+1},S_{t_{i+1}}\right)-G\left(t_i,S_{t_i}\right)\right|\leq M\left(\left(t_{i+1}-t_i\right)+\left|S_{i+1}-S_i\right|\right). \tag{6.40}$$

On the other hand, notice that for any real numbers A, B, and C, $||A+B|-C|\leq ||A|-C|+|B|$. Applying this result, as we write $F_{i+1}=F_{i+1}-F_i+F_i$ and

$$\begin{aligned}
&\int_{t_i}^{t_{i+1}} dsF\left(s,S_s\right)S_s\left|G_S\left(s,S_s\right)\right| \\
=&\int_{t_i}^{t_{i+1}} dsF\left(s,S_s\right)S_s\left|G_S\left(s,S_s\right)\right|-\left(t_{i+1}-t_i\right)F_iS_i\left|G_S\left(t_i,S_i\right)\right| \\
&+\left(t_{i+1}-t_i\right)F_iS_i\left|G_S\left(t_i,S_i\right)\right|
\end{aligned} \tag{6.41}$$

gives immediately

$$\begin{aligned}
&\left|\sum_{t_i,i\leq n}\sqrt{t_{i+1}-t_i}F_{i+1}\left|G_{i+1}-G_i\right|-\sqrt{\frac{2}{\pi}}\sigma\int_I dsF\left(s,S_s\right)S_s\left|G_S\left(s,S_s\right)\right|\right| \\
\leq&\left|\sum_{t_i,i\leq n}\sqrt{\Delta t}F_i\left|G_{i+1}-G_i\right|-\sum_{t_i,i\leq n}\sigma\sqrt{\frac{2}{\pi}}\Delta tF\left(t_i,S_i\right)S_i\left|G_S\left(t_i,S_i\right)\right|\right| \\
&+\left|\sum_{t_i,i\leq n}\sqrt{t_{i+1}-t_i}\left|F_{i+1}-F_i\right|\left|G_{i+1}-G_i\right|\right| \\
&+\sqrt{\frac{2}{\pi}}\sigma\sum_{t_i,i\leq n}\left|\int_{t_i}^{t_{i+1}} dsF\left(s,S_s\right)S_s\left|G_S\left(s,S_s\right)\right|-\left(t_{i+1}-t_i\right)F_iS_i\left|G_S\left(t_i,S_i\right)\right|\right. \\
&\qquad\left.\left(t_{i+1}-t_i\right)F_iS_i\left|G_S\left(t_i,S_i\right)\right|\right|.
\end{aligned} \tag{6.42}$$

Using now the different bounds we have determined in 6.38, 6.39, and 6.40, we obtain

$$\begin{aligned}
&\left|\sum_{t_i,i\leq n}\sqrt{t_{i+1}-t_i}F_{i+1}\left|G_{i+1}-G_i\right|-\sqrt{\frac{2}{\pi}}\sigma\int_I dsF\left(s,S_s\right)S_s\left|G_S\left(s,S_s\right)\right|\right| \\
\leq&\left|\sum_{t_i,i\leq n}\sqrt{t_{i+1}-t_i}F_i\left|G_{i+1}-G_i\right|-\sum_{t_i,i\leq n}\sigma\sqrt{\frac{2}{\pi}}\Delta tF\left(t_i,S_i\right)S_i\left|G_S\left(t_i,S_i\right)\right|\right| \\
&+M\sum_{t_i,i\leq n}\sqrt{t_{i+1}-t_i}\left|G_{i+1}-G_i\right|\left(\left(t_{i+1}-t_i\right)+\left|S_{i+1}-S_i\right|\right) \\
&+\sqrt{\frac{2}{\pi}}\sigma\sup_i\sup_{t_i\leq s\leq t_{i+1}}\left|F\left(s,S_s\right)S_s\left|G_S\left(s,S_s\right)\right|-F_iS_i\left|G_S\left(t_i,S_i\right)\right|\right|\left|I\right|.
\end{aligned} \tag{6.43}$$

For the third term in 6.43, we use the preceding upper bound

$$\begin{aligned}
&\sum_{t_i,i\leq n}\sqrt{t_{i+1}-t_i}\left|G_{i+1}-G_i\right|\left(\left(t_{i+1}-t_i\right)+\left|S_{i+1}-S_i\right|\right) \\
\leq\;& M\sum_{t_i,i\leq n}\left(t_{i+1}-t_i\right)^{\frac{5}{2}}+2M\sum_{t_i,i\leq n}\left|S_{i+1}-S_i\right|\sqrt{t_{i+1}-t_i}\left(t_{i+1}-t_i\right) \\
&+M\sup_i\left(\sqrt{t_{i+1}-t_i}\right)\int_I d\left[S\right].
\end{aligned} \tag{6.44}$$

We can also write the norm of this random variable, and obtain, if we forget multiplicative constants

$$
\begin{aligned}
& \mathbb{E}\left[\left|\sum_{t_i, i\leq n} \sqrt{t_{i+1}-t_i}\,|G_{i+1}-G_i|\left((t_{i+1}-t_i)+|S_{i+1}-S_i|\right)\right|\right] \\
\leq & \sum_{t_i, i\leq n} (t_{i+1}-t_i)^{\frac{5}{2}} + \mathbb{E}\left[\sum_{t_i, i\leq n} (t_{i+1}-t_i)^2 \sigma S_i \sqrt{\frac{2}{\pi}}\right] \\
& + \sup_i \left(\sqrt{t_{i+1}-t_i}\right) \mathbb{E}\int_I d\,[S]\,. \qquad (6.45)
\end{aligned}
$$

And these amounts are clearly converging to 0.

As for the fourth term in 6.43, we note first that since the functions F and G are continuous, then $F(s,S_s)\,S_s\,|G_S(s,S_s)|$ is continuous with respect to s. It is clear that in expectation

$$
\sup_i \sup_{t_i\leq s\leq t_{i+1}} \left|F(s,S_s)\,S_s\,|G_S(s,S_s)| - F_i S_i\,|G_S(t_i,S_i)|\right| \to 0. \qquad (6.46)
$$

It is now important to remark that for any series A and B and for any real C, we have

$$
\left|\sum |A_i+B_i| - C\right| \leq \left|\sum |A_i| - C\right| + \sum |B_i|\,. \qquad (6.47)
$$

Also noticing that

$$
\begin{aligned}
& \left|G\left(t_{i+1},S_{t_{i+1}}\right) - G\left(t_i,S_{t_i}\right)\right| \\
= & \left|G\left(t_{i+1},S_{t_{i+1}}\right) - G\left(t_i,S_{t_i}\right) - \Delta S_i G_S(t_i,S_i) + \Delta S_i G_S(t_i,S_i)\right| \qquad (6.48)
\end{aligned}
$$

we are in a position to write for the second term in 6.43

$$
\begin{aligned}
& \left|\sum_{t_i, i\leq n} \sqrt{t_{i+1}-t_i} F_i\,|G_{i+1}-G_i| - \sum_{t_i, i\leq n} \sigma\sqrt{\frac{2}{\pi}}\Delta t F(t_i,S_i)\,S_i\,|G_S(t_i,S_i)|\right| \\
\leq & \left|\sum_{t_i, i\leq n} \sqrt{t_{i+1}-t_i} F_i\,|\Delta S_i G_S(t_i,S_i)| - \sum_{t_i, i\leq n} \sigma\sqrt{\frac{2}{\pi}}\Delta t F(t_i,S_i)\,S_i\,|G_S(t_i,S_i)|\right| \\
& + \left|\sum_{t_i, i\leq n} \sqrt{t_{i+1}-t_i} F_i\left|G\left(t_{i+1},S_{t_{i+1}}\right) - G\left(t_i,S_{t_i}\right) - \Delta S_i G_S(t_i,S_i)\right|\right|. \qquad (6.49)
\end{aligned}
$$

The latter term can be bounded as follows

$$
\begin{aligned}
& \sum_{t_i, i\leq n} \sqrt{t_{i+1}-t_i} F_i\left|G\left(t_{i+1},S_{t_{i+1}}\right) - G\left(t_i,S_{t_i}\right) - \Delta S_i G_S(t_i,S_i)\right| \\
\leq & M\sqrt{\Delta t} \sum_{t_i, i\leq n} F_i\left(\Delta t + \Delta S_i^2\right). \qquad (6.50)
\end{aligned}
$$

Now, if we take the expectation and thus the norm, we can write

$$
\begin{aligned}
& \mathbb{E}\left[\sum_{t_i, i\leq n} \mathbb{E}_{\mathcal{F}_{t_i}} F_i\left(\Delta t + \Delta S_i^2\right)\right] \\
\leq & \mathbb{E}\left[\sum_{t_i, i\leq n} \Delta t F_i (1+M)\right] \qquad (6.51)
\end{aligned}
$$

and this amount converges to $\mathbb{E}\int_I F(s,S_s)\,ds$. Therefore, we have shown that

$$\sum_{t_i,i\le n}\sqrt{t_{i+1}-t_i}F_i\left|G\left(t_{i+1},S_{t_{i+1}}\right)-G\left(t_i,S_{t_i}\right)-\Delta S_i G_S\left(t_i,S_i\right)\right| \tag{6.52}$$

converges to 0 in L^1.

We are now interested in the remaining amount in 6.49, and will prove its convergence to 0. We note that

$$\begin{aligned}
&\left|\sum_{t_i,i\le n}\sqrt{t_{i+1}-t_i}F_i\left|\Delta S_i G_S\left(t_i,S_i\right)\right|-\sum_{t_i,i\le n}\sigma\sqrt{\frac{2}{\pi}}\Delta t F\left(t_i,S_i\right)S_i\left|G_S\left(t_i,S_i\right)\right|\right|\\
=&\left|\sum_{t_i,i\le n}\sqrt{\Delta t}F_i\left|G_S\left(t_i,S_i\right)\right|\left(\left|\Delta S_i\right|-\sigma\sqrt{\frac{2}{\pi}}\sqrt{\Delta t}S_i\right)\right|\\
=&\left|\sum_{t_i,i\le n}\sqrt{\Delta t}F_i\left|G_S\left(t_i,S_i\right)\right|\left(\left|\Delta S_i\right|-\mathbb{E}_{\mathcal{F}_{t_i}}\left|\Delta S_i\right|\right)\right|.
\end{aligned} \tag{6.53}$$

As the reader can check, it is not possible to show directly that this amount converges to 0 in expectation, and as a consequence we are going to consider the L^2 norm. We study

$$\begin{aligned}
&\mathbb{E}\left[\sum_{t_i,i\le n}\sqrt{\Delta t}F_i\left|G_S\left(t_i,S_i\right)\right|\left(\left|\Delta S_i\right|-\mathbb{E}_{\mathcal{F}_{t_i}}\left|\Delta S_i\right|\right)\right]^2\\
=&\ \mathbb{E}\left[\sum_{t_i,i\le n}\Delta t F_i^2\left|G_S\left(t_i,S_i\right)\right|^2\left(\left|\Delta S_i\right|-\mathbb{E}_{\mathcal{F}_{t_i}}\left|\Delta S_i\right|\right)^2\right]\\
&+\mathbb{E}\left[\sum_{t_j,j\le n,t_i,i\le j}\Delta t F_iF_j\left|G_S\left(t_i,S_i\right)\right|\left|G_S\left(t_j,S_j\right)\right|\right.\\
&\left.\left(\left|\Delta S_i\right|-\mathbb{E}_{\mathcal{F}_{t_i}}\left|\Delta S_i\right|\right)\left(\left|\Delta S_j\right|-\mathbb{E}_{\mathcal{F}_{tj}}\left|\Delta S_j\right|\right)\right]
\end{aligned} \tag{6.54}$$

As for the first term,

$$\begin{aligned}
&\mathbb{E}_{\mathcal{F}_{t_i}}\left(\left|\Delta S_i\right|-\mathbb{E}_{\mathcal{F}_{t_i}}\left|\Delta S_i\right|\right)^2\\
=&\ S_i^2\int\frac{dze^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\left(\left|e^{\left(\mu-\frac{\sigma^2}{2}\right)\Delta t+\sigma\sqrt{\Delta t}z}-1\right|-\sqrt{\Delta t}\sqrt{\frac{2}{\pi}}\sigma\right)^2
\end{aligned} \tag{6.55}$$

Thanks to the Mean Value theorem, we write

$$\begin{aligned}
&\left|e^{\left(\mu-\frac{\sigma^2}{2}\right)\Delta t+\sigma\sqrt{\Delta t}z}-1-\left(\mu-\frac{\sigma^2}{2}\right)\Delta t-\sigma\sqrt{\Delta t}z\right|\\
\le&\ M\Delta t^2\exp\left(M'z\right).
\end{aligned} \tag{6.56}$$

Therefore

$$\left|e^{\left(\mu-\frac{\sigma^2}{2}\right)\Delta t+\sigma\sqrt{\Delta t}z}-1-\sigma\sqrt{\Delta t}z\right|\le M\Delta t\exp\left(M'z\right). \tag{6.57}$$

First of all, let us write that for all A, B, C, D, if $|A-C|\leq D$ and D is positive, then

$$\left(|A|-B\right)^2\leq\left(|C|-B\right)^2+D^2+2D\left||C|-B\right|. \tag{6.58}$$

This makes it possible to state, simplifying the multiplicative constants

$$\begin{aligned}
&\left(\left|e^{\left(\mu-\frac{\sigma^2}{2}\right)\Delta t+\sigma\sqrt{\Delta t}z}-1\right|-\sqrt{\Delta t}\sqrt{\frac{2}{\pi}}\sigma\right) \\
\leq\ &\left(\left|\sigma\sqrt{\Delta t}z\right|-\sqrt{\Delta t}\sqrt{\frac{2}{\pi}}\sigma\right)^2+\sigma^2\Delta t z^2 \\
&+2\sqrt{\Delta t}\sqrt{\frac{2}{\pi}}\sigma\left|\left|\sigma\sqrt{\Delta t}z\right|-\sqrt{\Delta t}\sqrt{\frac{2}{\pi}}\sigma\right| \\
\leq\ &\Delta t M z^2.
\end{aligned} \tag{6.59}$$

Thus, we have

$$\mathbb{E}_{\mathcal{F}_{t_i}}\left(|\Delta S_i|-\mathbb{E}_{\mathcal{F}_{t_i}}|\Delta S_i|\right)^2\leq\Delta t M S_i^2\int\frac{dze^{-\frac{z^2}{2}}}{\sqrt{2\pi}}z^2\leq M\Delta t S_i^2. \tag{6.60}$$

This bounds help us write

$$\begin{aligned}
&\mathbb{E}\left[\sum_{t_i,i\leq n}\Delta t F_i^2\left|G_S\left(t_i,S_i\right)\right|^2\left(|\Delta S_i|-\mathbb{E}_{\mathcal{F}_{t_i}}|\Delta S_i|\right)^2\right] \\
\leq\ &M\Delta t\mathbb{E}\left[\sum_{t_i,i\leq n}\Delta t S_i^2F_i^2\left|G_S\left(t_i,S_i\right)\right|^2\right].
\end{aligned} \tag{6.61}$$

The expectation converges to $\mathbb{E}\int_I S_s^2F^2\left(s,S_s\right)\left|G_S\left(s,S_s\right)\right|^2ds$, which is assumed to be bounded, and therefore the whole quantity converges to 0, in $\mathbb{R}$.

As for the second term in 6.54, let us note that

$$\begin{aligned}
&\mathbb{E}\left[\sum_{t_j,j\leq n,t_i,i\leq j}\Delta t F_iF_j\left|G_S\left(t_i,S_i\right)\right|\left|G_S\left(t_j,S_j\right)\right|\right. \\
&\left.\left(|\Delta S_i|-\mathbb{E}_{\mathcal{F}_{t_i}}|\Delta S_i|\right)\left(|\Delta S_j|-\mathbb{E}_{\mathcal{F}_{tj}}|\Delta S_j|\right)\right] \\
=\ &\mathbb{E}\left[\sum_{t_j,j\leq n,t_i,i\leq j}\Delta t F_iF_j\left|G_S\left(t_i,S_i\right)\right|\left|G_S\left(t_j,S_j\right)\right|\right. \\
&\left.\mathbb{E}_{\mathcal{F}_{t_i}}\left[\left(|\Delta S_i|-\mathbb{E}_{\mathcal{F}_{t_i}}|\Delta S_i|\right)\left(|\Delta S_j|-\mathbb{E}_{\mathcal{F}_{tj}}|\Delta S_j|\right)\right]\right]
\end{aligned} \tag{6.62}$$

But

$$\begin{aligned}
&\mathbb{E}_{\mathcal{F}_{t_i}}\left[\left(|\Delta S_i|-\mathbb{E}_{\mathcal{F}_{t_i}}|\Delta S_i|\right)\left(|\Delta S_j|-\mathbb{E}_{\mathcal{F}_{tj}}|\Delta S_j|\right)\right] \\
=\ &\mathbb{E}_{\mathcal{F}_{t_i}}\left[|\Delta S_i|\,|\Delta S_j|\right]+\mathbb{E}_{\mathcal{F}_{t_i}}\left[\mathbb{E}_{\mathcal{F}_{t_i}}|\Delta S_i|\,\mathbb{E}_{\mathcal{F}_{tj}}|\Delta S_j|\right] \\
&-\mathbb{E}_{\mathcal{F}_{t_i}}\left[|\Delta S_i|\,\mathbb{E}_{\mathcal{F}_{tj}}|\Delta S_j|\right]-\mathbb{E}_{\mathcal{F}_{t_i}}\left[|\Delta S_j|\,\mathbb{E}_{\mathcal{F}_{t_i}}|\Delta S_i|\right].
\end{aligned} \tag{6.63}$$

Thanks to the independence of the variations of the Markov diffusion S, we can split the expectations and get

$$\begin{aligned}
& \mathbb{E}_{\mathcal{F}_{t_i}}\left[\left(|\Delta S_i| - \mathbb{E}_{\mathcal{F}_{t_i}}|\Delta S_i|\right)\left(|\Delta S_j| - \mathbb{E}_{\mathcal{F}_{tj}}|\Delta S_j|\right)\right] \\
= & \ \mathbb{E}_{\mathcal{F}_{t_i}}[|\Delta S_i|]\,\mathbb{E}_{\mathcal{F}_{t_i}}[|\Delta S_j|] + \quad \mathbb{E}_{\mathcal{F}_{t_i}}[|\Delta S_i|]\,\mathbb{E}_{\mathcal{F}_{t_i}}[|\Delta S_j|] \\
& -\mathbb{E}_{\mathcal{F}_{t_i}}[|\Delta S_i|]\,\mathbb{E}_{\mathcal{F}_{t_i}}[|\Delta S_j|] - \quad \mathbb{E}_{\mathcal{F}_{t_i}}[|\Delta S_i|]\ \mathbb{E}_{\mathcal{F}_{t_i}}[|\Delta S_j|] \\
= & \ 0. \qquad (6.64)
\end{aligned}$$

And this last result ends the proof. ■

We need to adapt a lemma shown by Henrotte about the convergence of a sum towards a stochastic integral.

Theorem 24 *Suppose that*

- $\eta^k(t, S_t)\frac{h_S(t,S_t)}{C_S^k(t,S_t)}$ *is* $C^{1,2}$ *with respect to* t *and* S_t
- F *and* G *are two* $C^{1,2}$ *functions*

Then we have the following convergence result for any k

$$\sup_j \left|t_{j+1}^k - t_j^k\right| \underset{a^k \to 0}{\to} 0 \ a.s.$$

and therefore

$$\sum_{t_j^k} F\left(t_j^k, S_{t_j^k}\right)\left(G\left(t_{j+1}^k, S_{j+1}^k\right) - G\left(t_j^k, S_{t_j^k}\right)\right) \underset{a^k \to 0}{\to} \int_0^T F(s, S_s)\, dG(s, S_s)$$

in probability over compacts.

Proof. We follow Henrotte's proof.

First, we recall the definition of t_j^k: $t_0^k = 0$ and

$$\begin{aligned}
t_{j+1}^k = \ & T \wedge \inf\left\{t \geq t_j^k : \eta(t, S_t)(dk)\frac{h_S(t,S_t)}{C_S^k(t,S_t)} = \right. \\
& \mathbb{I}_{\left|\eta(t_j,S_j)(dk)\frac{h_S(t_j,S_j)}{C_S^k(t_j,S_j)}\right| > \xi}\,\eta(t_j,S_j)(dk)\frac{h_S(t_j,S_j)}{C_S^k(t_j,S_j)}e^{-a^k} \\
& + \mathbb{I}_{\left|\eta(t_j,S_j)(dk)\frac{h_S(t_j,S_j)}{C_S^k(t_j,S_j)}\right| \leq \xi}\left(\eta(t_j,S_j)(dk)\frac{h_S(t_j,S_j)}{C_S^k(t_j,S_j)} + \xi\left(1 - e^{-a^k}\right)\right) \\
& \text{or } \mathbb{I}_{\left|\eta(t_j,S_j)(dk)\frac{h_S(t_j,S_j)}{C_S^k(t_j,S_j)}\right| > \xi}\,\eta(t_j,S_j)(dk)\frac{h_S(t_j,S_j)}{C_S^k(t_j,S_j)}e^{b^k} \\
& + \mathbb{I}_{\left|\eta(t_j,S_j)(dk)\frac{h_S(t_j,S_j)}{C_S^k(t_j,S_j)}\right| \leq \xi} \\
& \left.\left(\eta(t_j,S_j)(dk)\frac{h_S(t_j,S_j)}{C_S^k(t_j,S_j)} + \xi\left(e^{b^k} - 1\right)\right)\right\} \qquad (6.65)
\end{aligned}$$

Since it does not change anything, we will forget the k in the notations. We are interested in the study of $\Delta t_j^a = t_{j+1}^a - t_j^a$, where we note, instead of the k, the size a of the hedging band. Let us also recall that $1-e^{-a}=e^{b}-1$, and therefore $\frac{b}{a}$ converges to 1 when a goes to zero. If we also write

$$\eta\left(t_{j},S_{j}\right)(dk)\,\frac{h_{S}\left(t_{j},S_{j}\right)}{C_{S}^{k}\left(t_{j},S_{j}\right)}=U_{t_{j}^{a}} \tag{6.66}$$

then we can rewrite t_{j+1}^{a} in a simpler way:

$$\begin{aligned} t_{j+1}^{a} &= T\wedge\inf\left\{t\geq t_{j}^{a}:U_{t}=\mathbb{I}_{\left|U_{t_{j}^{a}}\right|>\xi}U_{t_{j}^{a}}e^{-a}\ +\mathbb{I}_{\left|U_{t_{j}^{a}}\right|\leq\xi}\left(U_{t_{j}^{a}}+\xi\left(1-e^{-a}\right)\right)\right.\\ &\quad \left.\text{or }\ \mathbb{I}_{\left|U_{t_{j}^{a}}\right|>\xi}U_{t_{j}^{a}}e^{b}+\mathbb{I}_{\left|U_{t_{j}^{a}}\right|\leq\xi}\left(U_{t_{j}^{a}}+\xi\left(e^{b}-1\right)\right)\right\}\wedge T\\ &= \mathbb{I}_{\left|U_{t_{j}^{a}}\right|>\xi}\inf\left\{t\geq t_{j}^{a}:U_{t}=U_{t_{j}^{a}}e^{-a}\text{ or }U_{t_{j}^{a}}e^{b}\right\}\wedge T\\ &\quad +\mathbb{I}_{\left|U_{t_{j}^{a}}\right|\leq\xi}\inf\left\{t\geq t_{j}^{a}:U_{t}=\left(U_{t_{j}^{a}}+\xi\left(1-e^{a}\right)\right)\ \text{ or }\ \left(U_{t_{j}^{a}}+\xi\left(e^{b}-1\right)\right)\right.\\ &\quad \left.\text{or }\ \left(U_{t_{j}^{a}}+\xi\left(e^{b}-1\right)\right)\right\} \end{aligned} \tag{6.67}$$

We note W_{a}^{u} the set of instants t_{j+1}^{a} such that $\left|U_{t_{j}^{a}}\right|>\xi$ and W_{a}^{d} the set of instants t_{j+1}^{a} such that $\left|U_{t_{j}^{a}}\right|\leq\xi$. We can write that $\sup_{j}\Delta t_{j}^{a}=\sup_{W_{a}^{u}}\Delta t_{j}^{a}\vee\sup_{W_{a}^{d}}\Delta t_{j}^{a}$. Therefore, we want to show that the set of trajectories A^{u} such that $\sup_{W_{a}^{u}}\Delta t_{j}^{a}$ does not converge to zero and the set of trajectories A^{d} such that $\sup_{W_{a}^{d}}\Delta t_{j}^{a}$ does not converge to zero are of Lebesgue measure zero. The procedure is to show that if these sets have not a zero measure, then for a as little as one wishes, it is possible to find a set of trajectories of S such that U does not go out of a band the size of which is determined by a; which is not consistent with the fact S is a diffusion.

Let us fix a trajectory in A^{u}. Since $\sup_{W_{a}^{u}}\Delta t_{j}^{a}$ does not converge to zero, there exists $M>0$ such that for all $\alpha>0$, there exists $a<\alpha$ such that $\sup_{W_{a}^{u}}\Delta t_{j}^{a}>M$. Using this fact, we can create a sequence (a_{n},b_{n}) converging to zero such that for all n $\sup_{W_{a}^{u}}\Delta t_{j}^{a}>M$. It is therefore possible to find a sequence of intervals $I_{n}=[c_{n},d_{n}]$ included in W_{a}^{u}, which set is included in $[0,T]$, such that $d_{n}-c_{n}>M$ and such that for all t,t' in I_{n}, $\left|\ln\left(|U_{t}|\right)-\ln\left(|U_{t'}|\right)\right|\leq a_{n}+b_{n}$. Since the sequence (c_{n},d_{n}) remains in the compact set $[0,T]\times[0,T]$, we can extract a converging subsequence with limit (c,d). We have $d-c\geq M$, and since (a_{n},b_{n}) is converging to zero, then $\ln(U)$ must be constant on $\left[c+\frac{M}{4},d-\frac{M}{4}\right]$. 4 is chosen, but we could have taken 3; the important thing being that there is a non-empty interval. But the set of trajectories where U is constant on an interval, because of the hypothesis that $\eta^{k}\frac{h_{S}}{C_{S}^{k}}$ is $\mathrm{C}^{1,2}$ as a function of (t,S_{t}), is negligeable. Indeed, Itô's theorem allows us to say that U is a diffusion. We conclude that A^{u} is included in a set of measure zero.

Now, let us follow the same procedure for A^{d}. We will only stress the differences with the arguments given above. We get that $|U_{t}-U_{t'}|\leq\xi\left(e^{b_{n}}-e^{-a_{n}}\right)$, and that U must be constant on a non-empty interval. The conclusion follows identically. Therefore A^{d} is negligeable.

Since $\sup_{W_{a}^{u}}\Delta t_{j}^{a}$ and $\sup_{W_{a}^{d}}\Delta t_{j}^{a}$ go to zero with a almost surely, we conclude that almost surely as well, $\sup_{j}\Delta t_{j}^{a}$ goes to zero.

Endowed with this last result, we are in a position to show the theorem, by following the lines of Lemma 3 in Henrotte (1994). A first consequence is that

$$\sup_{s\in[0,t]}\left|F(s,S_s)-\sum_j \mathbb{I}_{]t_j^a,t_{j+1}^a]}(s)\,F\left(t_j^a,S_{t_j^a}\right)\right| \tag{6.68}$$

converges almost surely to zero, by the almost sure uniform continuity of the trajectories of $F(t,S_t)$, being continuous over the compact set $[0,T]$. Therefore, $\left(\sum_j \mathbb{I}_{]t_j^a,t_{j+1}^a]}(s)\,F\left(t_j^a,S_{t_j^a}\right)\right)_{s\geq 0}$ converges to $(F(s,S_s))_{s\geq 0}$ uniformly on compacts a.s. and also in probability. But we can write

$$\begin{aligned} &\sum_{t_j^a} F\left(t_j^a,S_{t_j^a}\right)\left(G\left(t_{j+1}^a,S_{t_{j+1}^a}\right)-G\left(t_j^a,S_{t_j^a}\right)\right)\\ =\;& \int_0^T \sum_j \mathbb{I}_{]t_j^a,t_{j+1}^a]}(s)\,F\left(t_j^a,S_{t_j^a}\right)dG(s,S_s). \end{aligned} \tag{6.69}$$

Using Protter (1994), the right hand side of the equality converges in probability over compacts to $\int_0^T F(s,S_s)\,dG(s,S_s)$, and this ends the proof. ■

Applications to options pricing

We have seen there is some freedom in the choice of the products to use so as to hedge a derivative portfolio. Since the induced transaction costs are not necessarily the same depending on the products used for hedging, there is an interest in finding the cost-minimizing strategies. This can be especially true for products which have a long maturity, in which case cumulated transaction costs may be very expensive.

As it has been shown in the first section, the two equations giving the price of derivatives under transaction costs, with the correct boundary conditions, are:

$$BS(h)=\sqrt{\frac{2}{\pi}}\sigma\sum_k \tau^k C^k S\left|\eta_S^k\frac{h_S}{C_S^k}-\eta^k\frac{h_{SS}C_S^k-h_S C_{SS}^k}{\left(C_S^k\right)^2}\right| \tag{6.70}$$

for Leland's hedging strategy, and

$$\begin{aligned} BS(h) &= \sigma^2\sum_k \frac{S^2\tau^k C^k\left(\eta_S^k\frac{h_S}{C_S^k}+\eta^k\frac{h_{SS}C_S^k-h_S C_{SS}^k}{\left(C_S^k\right)^2}\right)^2}{\mathbb{I}_{\left|\eta^k\frac{h_S}{C_S^k}\right|>\xi}\left|\eta^k\frac{h_S}{C_S^k}\right|+\xi\mathbb{I}_{\left|\eta^k\frac{h_S}{C_S^k}\right|\leq\xi}}\\ &= \sigma^2 S^2\tau C\left(\frac{h_{SS}C_S-h_S C_{SS}}{C_S^2}\right)^2 \quad \text{under some conditions} \end{aligned} \tag{6.71}$$

for the risk-related hedging strategy. We write BS for the "Black-Scholes term" which usually intervenes in the pricing equation without transaction costs:

$$BS(h)=0.$$

Other terms accounting for the effect of transaction costs have been found; several of them are given in Dewynne et Al. (1995). For example, a limiting model of Davis et Al. (1993) has been proposed by Whalley and Willmott (1997) where the price of a derivative hedged with the underlying verifies

$$BS\left(h\right)=\frac{e^{-r(T-t)}}{\gamma}\left(\frac{3k_3\gamma^2S^4\sigma^3}{8e^{-2r(T-t)}}\right)^{\frac{2}{3}}\left(\left|h_{SS}-\frac{e^{-r(T-t)}\left(\mu-r\right)}{\gamma S^2\sigma^2}\right|\right)^{\frac{4}{5}}$$

where k_3 is the proportional cost depending on dollar volume, γ is risk-aversion, and μ is the drift of the underlying process under the real probability. Also, a model based on a hedging bandwidth of size $\frac{\varepsilon(S,h_S)}{S}$ around the delta has been proposed by Whalley and Willmott (1994). In this model, the price of the option verifies

$$BS\left(h\right)=\frac{\sigma^2S^4}{\varepsilon}\left(k_1+\left(k_2+k_3S\right)\frac{\sqrt{\varepsilon}}{S}\right)h_{SS}^2$$

with the transaction cost equal to $k_1+k_2N+k_3NS$, N being the number of shares transacted. Note that this model is not asymptotic and assumes the risk neutrality of the hedger with respect to hedging residuals. Also, it is very close to our risk-related hedging strategy, which also use a hedging bandwidth around the delta.

Existence of solutions to the pricing equations

As it has been documented in Dewynne et Al (1995) or in Avellaneda and Paras (1994), an equation such as 6.70 in the simplest case, with $k=1$, $C^1=S$, and $\eta^1=1$ can be ill-posed depending on the value of the *Leland Number* $\sqrt{\frac{2}{\pi}\frac{\tau}{\sigma}}$. In this case, a standard solution to the pricing equation cannot be found if hedging is performed the usual way. A price can be found, as a solution of an obstacle problem, and under some conditions hedging is not performed.

In our setting, as we can choose among various simple options to hedge, we have more flexibility. Let us find how we should choose these so that the pricing equation can be solved. We consider the case when only one option is used for hedging purposes.

Theorem 25 (Leland's Strategy Solution) *If the option C used for hedging verifies $\frac{h_{SS}}{h_S}\geq\frac{C_{SS}}{C_S}$, if the delta h_S has always the same sign as the option's delta C_S, and if the price C is positive then the pricing equation 6.70 possesses a unique classical solution in $C^{2,1}$.*

Note that the conditions required in the theorem are quite intuitive; they mean that the derivative used for hedging should have a delta which is less volatile than that of the option being hedged. Also, the closer the two quantities $\frac{h_{SS}}{h_S}$ and $\frac{C_{SS}}{C_S}$ are, the smaller the transaction costs.

Proof. We have

$$\frac{h_{SS}}{h_S}\geq\frac{C_{SS}}{C_S}\text{ so }\frac{h_{SS}C_S-h_SC_{SS}}{\left(C_S\right)^2}\geq 0.$$

Therefore the pricing equation simplifies to

$$BS\left(h\right)=\sqrt{\frac{2}{\pi}}\sigma\tau CS\frac{h_{SS}C_S-h_SC_{SS}}{\left(C_S\right)^2}$$

or

$$
\begin{aligned}
0 &= h_{SS}\left(\frac{\sigma^2}{2}S^2+\sqrt{\frac{2}{\pi}}\sigma\tau S\frac{C}{C_S}\right)\\
&\quad +h_S\left(rS-\sqrt{\frac{2}{\pi}}\sigma\tau SC\frac{C_{SS}}{(C_S)^2}\right)+h_t-rh.
\end{aligned}
$$

Since $\left(\frac{\sigma^2}{2}S^2+\sqrt{\frac{2}{\pi}}\sigma\tau S\frac{C}{C_S}\right)$ is always strictly positive, this equation is a linear parabolic PDE with terminal boundary conditions and therefore admits a classical solution for any continuous boundary conditions. ■

Now, we turn to the risk-related strategy, still in the simple case when only one option is used for hedging:

Theorem 26 (Risk-Related Strategy Solution) *If the option C used for hedging verifies*

$$
h_{SS}\leq\frac{1+4C\tau\frac{C_{SS}}{C_S^2}}{4C\tau\frac{C_S}{h_S}},
$$

and if the options deltas verify $h_S>\xi C_S$, for some ξ, then the pricing equation 6.71 possesses a classical solution.

Proof. We just have to show that the PDE is elliptic and degenerate in that case. The condition on the gamma h_{SS} entails that

$$
S^2\sigma^2\left(2\tau\frac{h_{SS}}{C_Sh_S}-2\tau\frac{C_{SS}}{C_S^2}-\frac{1}{2}\right)\leq 0. \tag{6.72}
$$

We can define the operator H by

$$
H\left(S,t,h\left(S,t\right),h_S,h_t,h_{SS}\right)=-BS(h)+\sigma S^2\tau C\left(\frac{h_{SS}C_S-h_SC_{SS}}{C_S^2}\right)^2.
$$

It corresponds to 6.71. Thanks to 6.72, the operator H verifies for all (S,t), h_S, h_t, h, and all f and g such that $f\geq g$

$$
H\left(S,t,h,h_S,h_t,f\right)\leq H\left(S,t,h,h_S,h_t,g\right).
$$

First, we show that h is a sub-solution of 6.71. For any (S_0,t_0), if we pick a test function ϕ on $C^{2,2}$, such that (S_0,t_0) is a local maxima of $h-\phi$, then we can assume without a loss of generality that $h\left(S_0,t_0\right)=\phi\left(S_0,t_0\right)$. Also, since (S_0,t_0) is a local maxima of $h-\phi$ and h, $\phi\in C^{2,2}$ then $h_{SS}\left(S_0,t_0\right)\leq\phi_{SS}\left(S_0,t_0\right)$ and $h_S\left(S_0,t_0\right)=\phi_S\left(S_0,t_0\right)$. As a result, we have

$$
\begin{aligned}
H\left(S_0,t_0,\phi\left(S_0,t_0\right),\phi_S,\phi_t,\phi_{SS}\right) &= H\left(S_0,t_0,h\left(S_0,t_0\right),h_S,h_t,\phi_{SS}\right)\\
&\leq H\left(S_0,t_0,h\left(S_0,t_0\right),h_S,h_t,\phi_{SS}\right)\\
&\leq 0.
\end{aligned}
$$

As a consequence, h is a sub-solution belonging to $C^{2,2}$ (as we assumed). A symmetrical argument allows to conclude that h is also a super-solution. Therefore, h is a classical solution of the equation. ■

Approximating the solution of the non-linear PDE

The preceding pricing models have supposed that the continuous cost due to transactions fees is financed by the portfolio. The replication strategy is therefore such that these fees are exactly paid by the portfolio. If we compared the limit price with that of Black and Scholes' model, corresponding to the PDE without the non-linear term, it would give the amount of money which, once invested along with the portfolio, is going to finance transaction costs.

We could as well imagine that the market-maker only asks for an average remuneration of his costs: banking on an averaging effect of his portfolio, he could ask for a "fair" price without transaction costs plus an additional amount equal to his expectation of transaction costs if he hedges his position with Black and Scholes model.

Let us consider the simplest case, that of Leland's standard model. Transaction costs are paid at a continuous rate corresponding to the absolute value of the gamma (the second derivative with respect to price). The equation writes

$$\begin{cases} P_t + \frac{1}{2}\sigma^2 x^2 P_{xx} + r\left(P - xP_x\right) = \sigma\sqrt{\frac{2}{\pi}}\tau x^2 \left|P_{xx}\right| \\ P\left(T,x\right) = f\left(x\right) \end{cases} \tag{6.73}$$

Feynman-Kac's theorem, which can be found in Karatzas and Shreve (1991), p.366, states that if a smooth enough function g is given, then if P satisfies the equation

$$\begin{cases} P_t + \frac{1}{2}\sigma^2 x^2 P_{xx} + r\left(P - xP_x\right) = \sigma\sqrt{\frac{2}{\pi}}\tau x^2 \left|g_{xx}\right| \\ P\left(T,x\right) = f\left(x\right) \end{cases} \tag{6.74}$$

we have

$$P\left(t,x\right) = BS\left(t,x\right) + \mathbb{E}_{S_t=x}\left[\int_t^T dse^{-rs}\sigma\sqrt{\frac{2}{\pi}}\tau S_s^2 \left|g_{xx}\left(s,S_s\right)\right|\right] \tag{6.75}$$

where the expectation is taken under the risk-neutral probability. If we assume now that g happens to be a solution X to the non-linear equation, we can write the following functional equation

$$X\left(t,x\right) = BS\left(t,x\right) + \mathbb{E}_{S_t=x}\left[\int_t^T dse^{-rs}\sigma\sqrt{\frac{2}{\pi}}\tau S_s^2 \left|X_{xx}\left(s,S_s\right)\right|\right]. \tag{6.76}$$

This result finally shows that the price of a path-independent product can be obtained using the path-dependent approach we defined earlier.

We intend to approximate equation 6.73 by a couple of linear equations

$$\begin{cases} U_t + \frac{1}{2}\sigma^2 x^2 U_{xx} + r\left(U - xU_x\right) = 0 \\ V_t + \frac{1}{2}\sigma^2 x^2 V_{xx} + r\left(V - xV_x\right) = \sigma\sqrt{\frac{2}{\pi}}\tau x^2 \left|U_{xx}\right| \\ U\left(T,x\right) = f\left(x\right) \\ V\left(T,x\right) = 0 \\ P^* = U + V \end{cases} \tag{6.77}$$

In fact, we suppose that the costs due to transactions are not paid as a function of the real rehedging policy, but as if the portfolio was rehedged using Black-Scholes model. The assumption is that the gamma in the transaction costs model is not too far from that of Black-Scholes' model.

Using Feynman-Kac's theorem, we also have that

$$P^* (0, x) = BS (0, x) + \mathbb{E}_{S_0=x} \left[\int_0^T dte^{-rt} \sigma \sqrt{\frac{2}{\pi}} \tau S_t^2 \left| BS_{xx} (t, S_t) \right| \right] \tag{6.78}$$

where the expectation is taken under risk-neutral probability. This expression has the great advantages to be computable as a double integral, and to be computable by simulation if there are complicated functionals intervening under the integral.

This method is in fact the first step of a convergent procedure. Let us define the operator F such that

$$Fu (t, x) = BS (t, x) + \mathbb{E}_{S_t=x} \left[\int_t^T dse^{-rs} \sigma \sqrt{\frac{2}{\pi}} \tau S_s^2 \left| u_{xx} (s, S_s) \right| \right]. \tag{6.79}$$

The limit of the sequence can be computed

$$\begin{aligned} P_0 &= BS + FBS \\ P_{n+1} &= BS + FP_n \end{aligned} \tag{6.80}$$

as we know that $P = BS + FP$.

Let us insist on the difference between viewing the first step of the procedure as an approximation or as a price calculated by the trader under the assumption of some sort of averaging effect of his portfolio. In the latter case, the trader will sell, or buy, the product at the price given by the model, but if he is consistent with his views, he will hedge it with the Black-Sholes delta. On the other hand, if he considers this first step as an approximation, he will use the derivative of the newly obtained price in state of the delta. He will be therefore closer to self-financing, whereas the former has no chance to self-finance his replication portfolio.

Reducing the set of possible replication strategies

Since all the prices we have computed so far potentially depend on the "allocation measure" which we define as a way of weighting the different redundant derivatives used to replicate the payoff, we can look for an optimization procedure. We consider that the trader's goal is to minimize his bid-ask spread. It is trivial that it corresponds to minimizing the "transaction costs" part of the price (the other part being the "fair value").

The difficulty is that the price depends on the strategy which has been chosen, and the optimal strategy depends on the price. Solving this problem demands that the price be written as an explicit functional of the strategy, and that the transaction costs intervening in this price be minimized with respect to the strategy.

Let us assume that we have a way of determining the price as a function of the strategy. It can be done using the approximation procedure we have described earlier, or by a direct resolution of the non-linear PDE. We have seen also that it is reasonable to put bounds on the different derivatives of η, these bounds representing a maximum speed of transaction.

In the case of the approximation procedure, if the "blind" strategy is used the total amount of transaction costs to be paid is

$$\begin{aligned} &\sqrt{\frac{2}{\pi}} \sigma \mathbb{E} \sum_i \int_0^T dse^{-rs} \tau^i (s, S_s) S_s C^i (s, S_s) \\ &\left| \eta_S^i \frac{h_S}{C_S^i} (s, S_s) + \eta^i \frac{h_{SS} C_S^i - h_S C_{SS}^i}{\left(C_S^i\right)^2} (s, S_s) \right| \end{aligned} \tag{6.81}$$

where h is the price of the derivative, assumed to be given. It would be ideal to be able to put all the weight at once on the cheapest product. The problem is that it induces an infinite cost to follow this strategy, because of the term η_S^i.

Let us define

$$\begin{aligned} A_t^i &= \left\{ x : \tau^i(t,x)\, x C^i(t,x) \left| \frac{h_{SS}C_S^i - h_S C_{SS}^i}{\left(C_S^i\right)^2}(t,x) \right| \right. \\ &\left. = \min_j \tau^j(t,x)\, x C^j(t,x) \left| \frac{h_{SS}C_S^j - h_S C_{SS}^j}{\left(C_S^j\right)^2}(t,x) \right| \right\}. \end{aligned} \tag{6.82}$$

The ideal would be, therefore, to have $\eta^i(t,x) = \mathbb{I}_{x \in A_t^i}$. But since it is not possible, we can choose to smooth this ideal function, depending on a parameter ε, the maximum slope for example. Then, we look for the optimal strategy among the set of these simpler strategies, which are easier to implement.

Concluding remarks

A model allowing a trader to hedge by means of various derivatives, in a path-dependent-case, has been presented. This model takes transaction costs into account. We have proposed a proof of the convergence of Leland's scheme towards a non-linear PDE in this general setting. As the setting includes transaction costs, an allocation strategy minimizing these costs can be followed.

It has clearly appeared there is a possibility to benefit from lower transactions fees by thoroughly choosing how to hedge a derivative. But the optimal strategy and the price of the hedged derivative are solutions of complicated equations, which we were only able to approach in simple cases. For example, if we consider usual path-dependent options, like barrier or lookback options, and contemplate hedging them with plain-vanilla options while minimizing cumulated transaction costs, the equation has to be solved numerically. The study of the quality of the approximations we proposed is also left for further research.

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Appendix: the path-dependent case

In this appendix we give the proof for the path-dependent case, which is a straight generalization of the path-independent case. We have the following result:

Theorem 27 *Let $\left(t_i^n, 0 \leq i \leq n\right)_{n \in \mathbb{N}}$ a series of uniform partitions of a positive interval I, F and G be two $C^{2,2,2}$ functions with bounded derivatives, F being positive, and S a continuous diffusion such that*

- $\lim_{n \to \infty} \sup_{i \leq n} \left(t_{i+1}^n - t_i^n\right) = 0$
- $dS_t = S_t \mu dt + S_t \sigma dB_t$ *where B is a Brownian Motion*
- *H is a finite variation process with continuous paths adapted to the natural filtration of S*
- $\mathbb{E} \int_I F\left(s, S_s, H_s\right) ds < \infty$ *and* $\mathbb{E} \int_I S_s^2 F^2\left(s, S_s, H_s\right) \left|G_S\left(s, S_s, H_s\right)\right|^2 ds < \infty$
- $\mathbb{E} V\left(H\right) = \mathbb{E} \sup_n \sum_{t_i} \left|H_{t_{i+1}} - H_{t_i}\right| < \infty$

Then we have the following convergence result:

$$\sum_{i=0}^{n}\sqrt{t_{i+1}^n-t_i^n}F\left(t_{i+1}^n,S_{t_{i+1}^n},H_{t_{i+1}^n}\right)\left|G\left(t_{i+1}^n,S_{t_{i+1}^n},H_{t_{i+1}^n}\right)-G\left(t_i^n,S_{t_i^n},H_{t_i^n}\right)\right|$$

converges in $L^1(\Omega,\mathbb{P})$ *to*

$$\sqrt{\frac{2}{\pi}}\int_I dsF\left(s,S_s,H_s\right)\sigma S_s\left|G_S\left(s,S_s,H_s\right)\right|.$$

Proof. We want to show the following quantity converges to zero

$$\mathbb{E}\left[\left|\sum_{t_i,i\leq n}\sqrt{t_{i+1}-t_i}F_{i+1}\left|G_{i+1}-G_i\right|-\sqrt{\frac{2}{\pi}}\int_I dsF\left(s,S_s,H_s\right)S_s\left|G_S\left(s,S_s,H_s\right)\right|\right|\right].$$

As in the demonstration of theorem 7.1, we study the random variable

$$\left|\sum_{t_i,i\leq n}\sqrt{t_{i+1}-t_i}F_{i+1}\left|G_{i+1}-G_i\right|-\sqrt{\frac{2}{\pi}}\int_I dsF\left(s,S_s,H_s\right)S_s\left|G_S\left(s,S_s,H_s\right)\right|\right|.$$

For a continuous function f, the following result still holds

$$\begin{aligned}
&\left|\int_{t_i}^{t_{i+1}}dsF\left(s,S_s,H_s\right)S_s\left|G_S\left(s,S_s,H_s\right)\right|-\left(t_{i+1}-t_i\right)F_iS_i\left|G_S\left(t_i,S_i,H_i\right)\right|\right|\\
\leq\;&\left(t_{i+1}-t_i\right)\sup_{t_i\leq s\leq t_{i+1}}\left|F\left(s,S_s,H_s\right)S_s\left|G_S\left(s,S_s,H_s\right)\right|-F_iS_i\left|G_S\left(t_i,S_i,H_i\right)\right|\right|.
\end{aligned}$$

If H had got jumps, we would not be in a position to show the convergence of this supremum to zero. Instead, we would have to add the hypothesis that $F\left(s,S_s,H_s\right)$ and $G_S\left(s,S_s,H_s\right)$ have continuous trajectories, or to find another way for the demonstration.

Using Taylor's theorem, we also have

$$\begin{aligned}
&\left|F\left(i+1\right)-F\left(i\right)-\Delta tF_t\left(i\right)-\Delta SF_S\left(i\right)-\Delta HF_H\left(i\right)\right|\\
\leq\;&\frac{\left(\Delta t^2+\Delta S^2+\Delta H^2\right)}{2}\sup_{y,z,t_i\leq s\leq t_{i+1}}\left(\left|F_{tt}\left(s,z,y\right)\right|+\left|F_{SS}\left(s,z,y\right)\right|+\left|F_{HH}\left(s,z,y\right)\right|+..\right)\\
\leq\;&M\left(\Delta^2+\Delta S^2+\Delta H^2\right)
\end{aligned}$$

because of the hypothesis that the derivatives are bounded. From this result we deduce that

$$\left|F\left(t_{i+1},S_{t_{i+1}},H_{t_{i+1}}\right)-F\left(t_i,S_{t_i},H_{t_i}\right)\right|\leq M\left(\Delta t+\left|\Delta S\right|+\left|\Delta H\right|\right)$$

as well as

$$\begin{aligned}
&\left|G\left(t_{i+1},S_{t_{i+1}},H_{t_{i+1}}\right)-G\left(t_i,S_{t_i},H_{t_i}\right)-\Delta S_iG_S\left(t_i,S_{t_i},H_{t_i}\right)\right|\\
\leq\;&M\left(\Delta t+\Delta S_i^2+\left|\Delta H\right|\right)
\end{aligned}$$

and

$$\left|G\left(t_{i+1},S_{t_{i+1}},H_{t_{i+1}}\right)-G\left(t_i,S_{t_i},H_{t_i}\right)\right|\leq M\left(\Delta t+\left|\Delta S\right|+\left|\Delta H\right|\right).$$

With the same procedure as before, we get

$$\begin{aligned}
&\left|\sum_{t_i,i\leq n}\sqrt{t_{i+1}-t_i}F_{i+1}\left|G_{i+1}-G_i\right|-\sqrt{\frac{2}{\pi}}\sigma\int_I dsF\left(s,S_s,H_s\right)S_s\left|G_S\left(s,S_s,H_s\right)\right|\right|\\
\leq\ &\left|\sum_{t_i,i\leq n}\sqrt{\Delta t}F_i\left|G_{i+1}-G_i\right|-\sum_{t_i,i\leq n}\sigma\sqrt{\frac{2}{\pi}}\Delta tF_iS_i\left|G_S\left(t_i,S_i,H_i\right)\right|\right|\\
&+\left|\sum_{t_i,i\leq n}\sqrt{t_{i+1}-t_i}\left|F_{i+1}-F_i\right|\left|G_{i+1}-G_i\right|\right|\\
&+\sqrt{\frac{2}{\pi}}\sigma\sum_{t_i,i\leq n}\left|\int_{t_i}^{t_{i+1}}dsF\left(s,S_s,H_s\right)S_s\left|G_S\left(s,S_s,H_s\right)\right|-\Delta tF_iS_i\left|G_S\left(t_i,S_i,H_i\right)\right|\right|.
\end{aligned}$$

Then we obtain easily

$$\begin{aligned}
&\left|\sum_{t_i,i\leq n}\sqrt{t_{i+1}-t_i}F_{i+1}\left|G_{i+1}-G_i\right|-\sqrt{\frac{2}{\pi}}\sigma\int_I dsF\left(s,S_s,H_s\right)S_s\left|G_S\left(s,S_s,H_s\right)\right|\right|\\
\leq\ &\left|\sum_{t_i,i\leq n}\sqrt{t_{i+1}-t_i}F_i\left|G_{i+1}-G_i\right|-\sum_{t_i,i\leq n}\sigma\sqrt{\frac{2}{\pi}}\Delta tF_iS_i\left|G_S\left(t_i,S_i,H_i\right)\right|\right| &(6.83)\\
&+M\sum_{t_i,i\leq n}\sqrt{t_{i+1}-t_i}\left|G_{i+1}-G_i\right|\left(\Delta t+\left|\Delta S\right|+\left|\Delta H\right|\right) &(6.84)\\
&+\sqrt{\frac{2}{\pi}}\sigma\sup_i\sup_{t_i\leq s\leq t_{i+1}}\left|F\left(s,S_s,H_s\right)S_s\left|G_S\left(s,S_s,H_s\right)\right|-F_iS_i\left|G_S\left(t_i,S_i,H_i\right)\right|\right.\\
&\left.-F_iS_i\left|G_S\left(t_i,S_i,H_i\right)\right|\right|\left|I\right|. &(6.85)
\end{aligned}$$

For 6.84, we use the preceding upper bounds

$$\sum_{t_i,i\leq n}\sqrt{t_{i+1}-t_i}\left|G_{i+1}-G_i\right|\left(\Delta t+\left|\Delta S\right|+\left|\Delta H\right|\right)\leq M\sum_{t_i,i\leq n}\sqrt{\Delta t}\left(\Delta t+\left|\Delta S\right|+\left|\Delta H\right|\right)^2$$

It is now possible to study the norm of this variable. We have already proved the convergence to zero of a part of this quantity, and we therefore prove it now only for the new terms.

$$\begin{aligned}
&\mathbb{E}\left[M\sum_{t_i,i\leq n}\left(\sqrt{\Delta t}\Delta H^2+\sqrt{\Delta t}\Delta t\left|\Delta H\right|+\sqrt{\Delta t}\left|\Delta S\right|\left|\Delta H\right|\right)\right]\\
\leq\ &M\sqrt{\Delta t}\mathbb{E}\left[\sum_{t_i,i\leq n}\Delta H^2+\sum_{t_i,i\leq n}\Delta t\left|\Delta H\right|+\sum_{t_i,i\leq n}\left|\Delta S\right|\left|\Delta H\right|\right]\\
\leq\ &\sqrt{\Delta t}\mathbb{E}\left[\sum_{t_i,i\leq n}\Delta H^2\right]+\sqrt{\Delta t}\Delta t\mathbb{E}\left[\sum_{t_i,i\leq n}\left|\Delta H\right|\right]+\sup_{\omega,t_i}\left|\Delta H\right|(\omega)\,\mathbb{E}\left[\sum_{t_i,i\leq n}\sqrt{\Delta t}\left|\Delta S\right|\right]
\end{aligned}$$

And these amounts are clearly converging to 0. Indeed, the first expectation is bounded since H is a finite variation process; for the same reason, and because of our hypotheses, the second expectation is also bounded. For the third expectation, we notice that the expectation has been shown to be bounded, and since H is continuous, the supremum goes to zero.

As for 6.85, the convergence to zero comes from the continuity of the processes under consideration, as in the proof of Theorem 23.

We are now interested in 6.83. We write

$$|G_{i+1} - G_i| = |G_{i+1} - G_i - \Delta S_i G_S + \Delta S_i G_S|$$

and then obtain

$$\begin{aligned} & \left| \sum_{t_i, i \leq n} \sqrt{\Delta t} F_i \left| G_{i+1} - G_i \right| - \sum_{t_i, i \leq n} \sigma \sqrt{\frac{2}{\pi}} \Delta t F_i S_i \left| G_S\left(t_i, S_i, H_i\right) \right| \right| \\ \leq & \left| \sum_{t_i, i \leq n} \sqrt{\Delta t} F_i \left| \Delta S_i G_S \right| - \sum_{t_i, i \leq n} \sigma \sqrt{\frac{2}{\pi}} \Delta t F_i S_i \left| G_S\left(t_i, S_i, H_i\right) \right| \right| \\ & + \left| \sum_{t_i, i \leq n} \sqrt{\Delta t} F_i \left| G_{i+1} - G_i - \Delta S_i G_S \right| \right|. \end{aligned}$$

The latter term can be bounded as follows:

$$\begin{aligned} & \sum_{t_i, i \leq n} \sqrt{\Delta t} F_i \left| G_{i+1} - G_i - \Delta S_i G_S \right| \\ \leq \ & M \sqrt{\Delta t} \sum_{t_i, i \leq n} F_i \left(\Delta t + \Delta S_i^2 + |\Delta H| \right). \end{aligned}$$

Now, if we take the expectation and thus the norm, we can write

$$\begin{aligned} & \mathbb{E}\left[\sum_{t_i, i \leq n} \mathbb{E}_{\mathcal{F}_{t_i}} F_i \left(\Delta t + \Delta S_i^2 + |\Delta H| \right) \right] \\ \leq \ & \mathbb{E}\left[\sum_{t_i, i \leq n} \mathbb{E}_{\mathcal{F}_{t_i}} F_i \left(\Delta t + \Delta S_i^2 \right) \right] + \mathbb{E}\left[\sum_{t_i, i \leq n} F_i \left| \Delta H \right| \right] \\ \leq \ & \mathbb{E}\left[\sum_{t_i, i \leq n} \Delta t F_i \left(1 + M \right) \right] + \mathbb{E}\left[\sum_{t_i, i \leq n} F_i \left| \Delta H \right| \right] \end{aligned}$$

and this amount converges to $\mathbb{E}\left[\int_I F\left(s, S_s\right) |dH_s| + \int_I F\left(s, S_s, H_s\right) \right] ds$, staying positive. Therefore, we have shown that

$$\sum_{t_i, i \leq n} \sqrt{\Delta t} F_i \left| G_{i+1} - G_i - \Delta S_i G_S \right|$$

converges to 0 in L^1.

We are now interested in the remaining amount of the expression, and will prove its convergence to 0, using extensively the procedure we followed to show Theorem 23

$$\begin{aligned} & \left| \sum_{t_i, i \leq n} \sqrt{\Delta t} F_i \left| \Delta S_i G_S \right| - \sum_{t_i, i \leq n} \sigma \sqrt{\frac{2}{\pi}} \Delta t F_i S_i \left| G_S \right| \right| \\ = & \left| \sum_{t_i, i \leq n} \sqrt{\Delta t} F_i \left| G_S\left(t_i, S_i, H_i\right) \right| \left(\left| \Delta S_i \right| - \mathbb{E}_{\mathcal{F}_{t_i}} \left| \Delta S_i \right| \right) \right|. \end{aligned}$$

As before, we consider the L^2 norm. We study

$$
\begin{aligned}
& \mathbb{E}\left[\sum_{t_i, i \leq n} \sqrt{\Delta t} F_i \left|G_S\right| \left(\left|\Delta S_i\right| - \mathbb{E}_{\mathcal{F}_{t_i}} \left|\Delta S_i\right|\right)\right]^2 \\
= \; & \mathbb{E}\left[\sum_{t_i, i \leq n} \Delta t F_i^2 \left|G_S\right|^2 \left(\left|\Delta S_i\right| - \mathbb{E}_{\mathcal{F}_{t_i}} \left|\Delta S_i\right|\right)^2\right] \\
& + \mathbb{E}\left[\sum_{t_j, j \leq n, t_i, i \leq j} \Delta t F_i F_j \left|G_S\left(t_i, S_i, H_i\right)\right| \left|G_S\left(t_j, S_j, H_j\right)\right|\right. \\
& \left. \times \left(\left|\Delta S_i\right| - \mathbb{E}_{\mathcal{F}_{t_i}} \left|\Delta S_i\right|\right) \left(\left|\Delta S_j\right| - \mathbb{E}_{\mathcal{F}_{tj}} \left|\Delta S_j\right|\right)\right].
\end{aligned}
$$

Then, with the help of the results we have already obtained, we get

$$
\begin{aligned}
& \mathbb{E}_{\mathcal{F}_{t_i}} \left(\left|\Delta S_i\right| - \mathbb{E}_{\mathcal{F}_{t_i}} \left|\Delta S_i\right|\right)^2 \\
\leq \; & M \Delta t S_i^2.
\end{aligned}
$$

The expectation converges to $\mathbb{E} \int_I S_s^2 F^2\left(s, S_s, H_s\right) \left|G_S\left(s, S_s, H_s\right)\right|^2 ds$, which is assumed to be bounded, and therefore the whole quantity converges to 0, in $\mathbb{R}$. For the cross-product term, the same procedure allows us to get the expected convergence result. And therefore, everything converges to zero ■

As for the extension of Theorem 24 to the path-dependent case, an added dependence on the past price evolution will not change a lot of things. Indeed, it will only add to the "volatility" of the process $\eta^k \frac{h_S}{C_S^k} = U$ and the rehedging instants will be even tighter.

Chapter 7 HEDGING ENTRY AND EXIT DECISIONS[1]

Real options theory establishes an analogy between monopolistic investment projects and financial options. An investment project, as we have seen in the previous chapters, contains an option to wait for a better time to invest, depending on the evolution of the random variables that condition the project's profitability (such as market share, commodity prices or labor costs). The classical Net Present Value rule, which prescribes investment as soon as a positive value is generated today, does not take into account the value contained in the option to wait. The option to exit an investment that has proven unattractive also possesses a value. Entry or exit are in most cases costly, that is there is an often important fixed cost in stopping a manufacturing plant, or restarting it.

Under these conditions, it is well known that the optimal strategy in terms of the entry or exit from a perpetual investment opportunity, is composed of two levels that will respectively trigger investment and disinvestment, when hit by the relevant variable. Dixit and Pindyck (1994) have presented an extensive survey of real option models, and of the entry/exit decision in particular. The value of an investment, as far as its management follows this rational decision behavior, is a function of the index level (the level of the relevant variables), and depends on whether the project is activated or not. If the underlying variable upon which the future cashflows of the project depend is traded in the marketplace, the value of the option can be determined thanks to a no arbitrage principle[2]. If it is not traded, one can resort to assuming investors are risk-neutral, and the actual calculation of the project value remains the same, with only a difference in the drift of the underlying variable. This risk-neutral assumption, and how some aversion of risk can be factored into the risk-free rate, is discussed in Trigeorgis (1996).

In practice, the levels at which a firm will enter or exit an investment are not necessarily optimal. McDonald (1999) has shown that there are "rules of thumb" used by corporations managers that allow them to implicitly proxy real options, such as hurdle rates or profitability requirements. This does not mean that many firms are poorly run, but rather that these proxy strategies do a good job of improving the straight Net Present Value rule. One consequence of this observation is that different investors might want to apply different entry and exit levels to the same project, even though they have the same entry costs and underlying dynamics. Different managers may also be able to lower entry or exit costs, or to change the dynamics of the project's stream of chas-flows, which would result in different optimal entry and exit levels.

Our goal in this chapter is to see how the buyers of a business could "hedge" the difference between their own preferred "optimal" levels and the ones actually in place (whether the difference in the levels comes from a more optimal strategy

[1] THIS CHAPTER HAS BEEN PUBLISHED IN THE J*OURNAL OF APPLIED MATHEMATICS AND DECISION SCIENCES* (ERLBAUM), 6(1), PP. 51-70 (2002), UNDER THE TITLE "HEDGING ENTRY AND EXIT DECISIONS: ACTIVATING AND DEACTIVATING BARRIER OPTIONS".

[2] As in the case of financial options, any future stream of cashflows that can be replicated can be priced, and its value equals the expectation of these cashflows under the risk-neutral probability. See Dixit and Pindyck (1994) or Trigeorgis (1996) for an exposition of these principles applied to real options.

or from different costs). In our analysis, we introduce a new kind of derivatives, which we call Switch options, and show that they are a well-adapted instrument to hedge the risk related to business entry and exit decisions. We will see how Switch options allow the buyer of a business to cancel the risk that, when he implements his new entry/exit strategy (as opposed to the prior strategy followed by the previous owner), he may have to pay immediate entry or exit costs. Since these entry or exit costs depend on whether the firm is active or not when he implements the new strategy, then the buyer incurs a random cost. The larger the delay between the buying of the business and when the buyer is actually able to implement his new strategy, the greater the potential discrepancy between the actual activity level of the firm and the optimal activity level.

We define Switch options as path-dependent derivatives written on a single underlying that are activated every time the underlying hits a barrier and deactivated every time it hits another barrier. At maturity, if the option is activated, the holder receives a payoff that is a function of the underlying at that time; if it is not activated, the payoff is a different and lower function of the underlying's price. The number of times such an option can be activated and deactivated is not bounded. Unlike a standard barrier option, the Switch option is never totally cancelled when the underlying hits the barrier, as there is always a chance it will go back and hit the other barrier. To our knowledge, such options are not currently traded with significant volume in financial markets.

The first section of the chapter focuses on a simple probabilistic approach to calculate the value of an investment with given entry and exit thresholds. In the second section, we give a general pricing formula for Switch options and compare them with standard barrier options. In the third section, we analyze the relationship between real options and Switch options, and show how the latter can help hedge the former. The Fourth section contains a proof for a central theorem in the second section's approach. This proof makes use of the Brownian Meander. Finally, the fifth section concludes the chapter.

Real options: entry and exit decisions

The optimal barriers that determine entry or exit decision can be found, and are thoroughly studied in Dixit and Pindyck's book, as well as in the important academic literature on the subject[3]. We propose here a simple probabilistic approach to derive a closed-form solution for the value of an investment with entry and exit. Brennan and Schwartz (1985) proposed a numerical method to calculate the value of an investment project with entry and exit, when the entry and exit levels are determined optimally. Our goal here is to write the value of the investment project **as a function** of the entry and exit barriers.

If we derive the value of an investment, dependent on the entry and exit levels a and d, then, the optimal value of this investment will be its maximum with respect to both a and b.

We assume that there is an underlying economic variable S, driven by the following process:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t,$$

where B is a Brownian Motion on the measured space $(\Omega, \mathcal{F}, \mathbb{P})$, and $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration of B. S represents the future stream of cash-flows (revenues

[3] Dixit and Pindyck's book contains an extensive review of the literature.

minus current expenses) generated by the investment project, and could therefore be dependent on oil prices (if the project is a refinery), gold prices (in the case of a gold mine), or even on the market share in a given product. As in most of the real-option theory, let us consider that the underlying variable, or one of its derivatives, is traded.

Since there is only one source of noise, the market is complete. μ is assumed to be smaller than the risk-free rate r used for discounting the cashflows. Equivalently, we can assume that $\mu = r - \delta$, where δ is the convenience yield (a continuous dividend paid by the project), and reflects a degree of risk aversion of investors. We suppose there is a running cost of c, so that the stream of cashflows generated by the project may sometimes becomes negative. For example, this constant running cost could be the recurrent expenses the firm has to pay to maintain the activity, which are not proportional to profits, such as payroll expenses.

In this case, we easily derive that the value of a perpetual project at time 0 is

$$\begin{aligned} \mathbb{E}_{S_0}\left[\int_0^\infty dse^{-rs}\left(S_s - c\right)\right] &= \frac{S_0}{\delta} - \frac{c}{r} \\ &= F_\infty\left(S_0\right) - \frac{c}{r} \end{aligned}$$

where F_∞ represents the gross revenue from the project.

Lemma 28 *The value F of a perpetual investment project that can be entered at level a and exited at level d, for the respective costs of C_e and C_x verifies:*

$$\begin{aligned} F\left(d\right) &= \frac{\left(\frac{d}{a}\right)^{\sqrt{\frac{2}{\sigma^2}\left(r+\frac{\xi^2}{2}\right)}-\frac{\xi}{\sigma}}\left(\frac{a}{r-\mu} - \frac{c}{r} - C_e\right) - \left(\frac{d}{a}\right)^{2\sqrt{\frac{2}{\sigma^2}\left(r+\frac{\xi^2}{2}\right)}}\left(\frac{d}{r-\mu} + C_x - \frac{c}{r}\right)}{1 - \left(\frac{d}{a}\right)^{2\sqrt{\frac{2}{\sigma^2}\left(r+\frac{\xi^2}{2}\right)}}} \\ F\left(a\right) &= \frac{\frac{a}{r-\mu} - \frac{c}{r} - \left(\frac{d}{a}\right)^{\sqrt{\frac{2}{\sigma^2}\left(r+\frac{\xi^2}{2}\right)}+\frac{\xi}{\sigma}}\left(\frac{d}{r-\mu} + C_x - \frac{c}{r}\right) - \left(\frac{d}{a}\right)^{2\sqrt{\frac{2}{\sigma^2}\left(r+\frac{\xi^2}{2}\right)}}C_e}{1 - \left(\frac{d}{a}\right)^{2\sqrt{\frac{2}{\sigma^2}\left(r+\frac{\xi^2}{2}\right)}}}, \end{aligned}$$

where $\xi = \frac{1}{\sigma}\left(\mu - \frac{\sigma^2}{2}\right)$.

Proof. We write the value of the investment opportunity just when it has been deactivated as a function of its value just when it has been activated, and reciprocally. Naturally, the value of the investment does not depend on time, only on the level of the underlying variable and whether it is active or not. Consequently, the optimal strategy is time-invariant, meaning that it can be represented by the constant levels a and d. Using the strong Markov property of the Brownian Motion, which ensures that Brownian increments before and after a stopping time are independent, gives us:

$$\begin{aligned} F\left(d\right) &= \mathbb{E}_d\left[e^{-rT_a(S)}\right]\left(F\left(a\right) - C_e\right) \\ F\left(a\right) &= \mathbb{E}_a\left[\int_0^{T_d(S)} dse^{-rs}\left(S_s - c\right)\right] + \mathbb{E}_a\left[e^{-rT_d(S)}\right]\left(F\left(d\right) - C_x\right) \\ &= \mathbb{E}_a\left[\int_0^\infty dse^{-rs}S_s\right] - \mathbb{E}_a\left[\int_{T_d(S)}^\infty dse^{-rs}S_s\right] \end{aligned}$$

$$
\begin{aligned}
& -\frac{c}{r}\left(1-\mathbb{E}_a\left[e^{-rT_d(S)}\right]\right)+\ \mathbb{E}_a\left[e^{-rT_d(S)}\right](F(d)-C_x) \\
= & \left(F_\infty(a)-\frac{c}{r}\right)-\mathbb{E}_a\left[e^{-rT_d(S)}\right]\left(F_\infty(d)+C_x-\frac{c}{r}\right)+\mathbb{E}_a\left[e^{-rT_d(S)}\right]F(d).
\end{aligned}
$$

By solving for $F(a)$ and $F(d)$ we get

$$
\begin{aligned}
F(d) &= \frac{\mathbb{E}_d\left[e^{-rT_a(S)}\right]\left(F_\infty(a)-\frac{c}{r}\right)-\mathbb{E}_d\left[e^{-rT_a(S)}\right]\mathbb{E}_a\left[e^{-rT_d(S)}\right]F_\infty(d)}{1-\mathbb{E}_d\left[e^{-rT_a(S)}\right]\mathbb{E}_a\left[e^{-rT_d(S)}\right]} \\
&\quad -\frac{\mathbb{E}_d\left[e^{-rT_a(S)}\right]\mathbb{E}_a\left[e^{-rT_d(S)}\right]\left(C_x-\frac{c}{r}\right)+\mathbb{E}_d\left[e^{-rT_a(S)}\right]C_e}{1-\mathbb{E}_d\left[e^{-rT_a(S)}\right]\mathbb{E}_a\left[e^{-rT_d(S)}\right]} \\
F(a) &= \frac{F_\infty(a)-\frac{c}{r}-\mathbb{E}_a\left[e^{-rT_d(S)}\right]F_\infty(d)}{1-\mathbb{E}_d\left[e^{-rT_a(S)}\right]\mathbb{E}_a\left[e^{-rT_d(S)}\right]} \\
&\quad -\frac{\mathbb{E}_d\left[e^{-rT_a(S)}\right]\mathbb{E}_a\left[e^{-rT_d(S)}\right]C_e+\mathbb{E}_a\left[e^{-rT_d(S)}\right]\left(C_x-\frac{c}{r}\right)}{1-\mathbb{E}_d\left[e^{-rT_a(S)}\right]\mathbb{E}_a\left[e^{-rT_d(S)}\right]}.
\end{aligned}
$$

We write for $a \geq S_0$

$$
\mathbb{E}_{S_0}\left[e^{-rT_a(S)}\right]=\mathbb{E}_0\left[e^{-rT_{\frac{1}{\sigma}\ln\left(\frac{a}{S_0}\right)}\left(B^\xi\right)}\right]
$$

where $\xi=\frac{1}{\sigma}\left(\mu-\frac{\sigma^2}{2}\right)$ and $\left(B_t^\xi\right)_{t\geq 0}=(B_t+\xi t)_{t\geq 0}$ is the drifted Brownian Motion. We used the fact that

$$
\begin{aligned}
T_a(S) &= \inf\{t\geq 0, S_t=a\} \\
&= \inf\left\{t\geq 0, S_0\exp\left(\sigma B_t^\xi\right)=a\right\} \\
&= \inf\left\{t\geq 0, B_t^\xi=\frac{1}{\sigma}\ln\left(\frac{a}{S_0}\right)\right\}.
\end{aligned}
$$

Now, we use the well-known fact that for a Brownian Motion, $\mathbb{E}\left[e^{-\lambda T_h}\right]=e^{-|h|\sqrt{2\lambda}}$, combined with Girsanov's theorem, and we get:

$$
\begin{aligned}
\mathbb{E}_{S_0}\left[e^{-rT_a(S)}\right] &= \mathbb{E}_0\left[e^{\xi B_{T_{\frac{1}{\sigma}\ln\left(\frac{a}{S_0}\right)}}-\frac{\xi^2}{2}T_{\frac{1}{\sigma}\ln\left(\frac{a}{S_0}\right)}}e^{-rT_{\frac{1}{\sigma}\ln\left(\frac{a}{S_0}\right)}(B)}\right] \\
&= \mathbb{E}_0\left[e^{\xi\frac{1}{\sigma}\ln\left(\frac{a}{S_0}\right)}e^{-\left(r+\frac{\xi^2}{2}\right)T_{\frac{1}{\sigma}\ln\left(\frac{a}{S_0}\right)}(B)}\right] \\
&= \left(\frac{S_0}{a}\right)^{\sqrt{\frac{2}{\sigma^2}\left(r+\frac{\xi^2}{2}\right)}-\frac{\xi}{\sigma}}. \qquad (7.1)
\end{aligned}
$$

Following the same approach for $d\leq S_0$ gets us

$$
\mathbb{E}_{S_0}\left[e^{-rT_d(S)}\right]=\left(\frac{d}{S_0}\right)^{\sqrt{\frac{2}{\sigma^2}\left(r+\frac{\xi^2}{2}\right)}+\frac{\xi}{\sigma}}. \qquad (7.2)
$$

These calculations allow us to write

$$
\begin{aligned}
F(d) &= \frac{\left(\frac{d}{a}\right)^{\sqrt{\frac{2}{\sigma^2}\left(r+\frac{\xi^2}{2}\right)}-\frac{\xi}{\sigma}}\left(\frac{a}{r-\mu}-\frac{c}{r}-C_e\right)-\left(\frac{d}{a}\right)^{2\sqrt{\frac{2}{\sigma^2}\left(r+\frac{\xi^2}{2}\right)}}\left(\frac{d}{r-\mu}+C_x-\frac{c}{r}\right)}{1-\left(\frac{d}{a}\right)^{2\sqrt{\frac{2}{\sigma^2}\left(r+\frac{\xi^2}{2}\right)}}} \\
F(a) &= \frac{\frac{a}{r-\mu}-\frac{c}{r}-\left(\frac{d}{a}\right)^{\sqrt{\frac{2}{\sigma^2}\left(r+\frac{\xi^2}{2}\right)}+\frac{\xi}{\sigma}}\left(\frac{d}{r-\mu}+C_x-\frac{c}{r}\right)-\left(\frac{d}{a}\right)^{2\sqrt{\frac{2}{\sigma^2}\left(r+\frac{\xi^2}{2}\right)}}C_e}{1-\left(\frac{d}{a}\right)^{2\sqrt{\frac{2}{\sigma^2}\left(r+\frac{\xi^2}{2}\right)}}},
\end{aligned}
$$

which is the result from the lemma. ■

We now have the following

Proposition 29 *The value of an investment project verifies*

$$
\begin{aligned}
F_{deact}(S_0) &= \left(\frac{S_0}{a}\right)^{\sqrt{\frac{2}{\sigma^2}\left(r+\frac{\xi^2}{2}\right)}-\frac{\xi}{\sigma}}\left(F(a)-C_e\right) \\
F_{act}(S_0) &= \left(\frac{d}{S_0}\right)^{\sqrt{\frac{2}{\sigma^2}\left(r+\frac{\xi^2}{2}\right)}+\frac{\xi}{\sigma}}\left(F(d)-C_x-\frac{d}{r-\mu}+\frac{c}{r}\right)+\frac{S_0}{r-\mu}-\frac{c}{r}.
\end{aligned}
$$

Proof. If the project is not activated, then its value is the present value of its value when it is activated, minus the cost of activation:

$$
F_{deact}(S_0) = \mathbb{E}_{S_0}\left[e^{-rT_a(S)}\right]\left(F(a)-C_e\right).
$$

If the project is activated, then its value is the present value of the cashflows it will generate until it is stopped, plus its present value deactivated, minus the cost of deactivation:

$$
\begin{aligned}
F_{act}(S_0) &= \mathbb{E}_{S_0}\left[\int_0^{T_d(S)} dse^{-rs}\left(S_s-c\right)\right]+\mathbb{E}_{S_0}\left[e^{-rT_d(S)}\right]\left(F(d)-C_x\right) \\
&= \mathbb{E}_{S_0}\left[e^{-rT_d(S)}\right]\left(F(d)-C_x\right)+F_\infty(S_0)-\frac{c}{r}-\mathbb{E}_{S_0}\left[e^{-rT_d(S)}\right]\left(F_\infty(d)-\frac{c}{r}\right).
\end{aligned}
$$

We obtain the result by replacing in the above expressions the value for 7.1 and 7.2. ■

The value of an investment project can be maximized with respect to the entry and exit levels a and d. Let us look at an example, where we make the following assumptions:

$$
\begin{aligned}
C_e &= 500, C_x = 50, c = 93.3 \\
r &= 8\%, \mu = 2\%, \sigma = 20\% \text{ and } S_0 = 100.
\end{aligned}
$$

In these conditions, the value of the project if it is started right away is about 0 (after entry costs). If the project is running and if it is never deactivated, its value is about 500. The following tables show the project's value, depending on the entry levels a and d. Note that apart from the case where a or d are very close to S_0, the project value is not very sensitive to the specific choice of barriers. This would seem to confirm the opinion developed in MacDonald (1999). It appears

Active value	100	110	120	130	140	150	160	170	180
100	NA	-288	-84	53	147	212	257	288	309
90	-14	141	243	312	360	392	415	430	440
80	305	379	429	463	486	502	513	520	525
70	470	505	529	545	556	564	569	572	574
60	548	564	575	583	588	591	594	596	597
50	576	583	588	591	594	595	596	597	597
40	574	577	579	581	582	582	583	583	583
30	556	557	558	558	559	559	559	559	559
20	532	532	532	532	532	532	532	532	532

Table 7.1 Valuation of an Active Project

Inactive value	100	110	120	130	140	150	160	170	180
100	NA	-520	-201	0	131	218	277	317	345
90	-638	-259	-28	119	216	280	324	353	372
80	-343	-72	96	205	277	325	357	379	392
70	-144	53	180	263	318	355	380	396	406
60	-23	130	230	297	343	373	393	406	414
50	40	169	255	314	355	382	400	411	420
40	60	180	262	319	357	384	401	412	421
30	52	174	257	315	354	381	399	410	420
20	31	159	247	307	348	377	396	408	418

Table 7.2 Valuation of an Inactive Project

that as far as the exit level is below 70, the value of the active project is greater by following an entry/exit strategy than by just leaving the project in place (the values in the table are above 500).

Table 7.1 on p. 124 shows the active value; the different columns represent various inputs for a (above the current index level of 100), while the rows represent various inputs for d (below 100). Table 7.2 on p. 124 shows the value for an inactive project.

The pricing of switch options

Switch options cannot be constituted out of finite combinations of single or double barrier options. A Switch option cannot either be valued using a two-dimensional partial differential equation, the problem being that the value at a barrier depends on whether it has been activated or deactivated, and therefore depends on its value at the preceding barrier hitting time.

These products have a payoff that is a function of

- the underlying value at maturity,
- whether or not the option is active at maturity, which depends on whether the underlying value has crossed an activating or deactivating barrier.

The latter feature can be summarized as: whether the underlying value has crossed a deactivating barrier since it last crossed an activating barrier, or since it was first activated.

So as to price options, the Black-Scholes model standardly assumes, among other things, that it is possible to perfectly replicate the payoff of a derivative product with a self-financing portfolio. If there is only one source of risk, it implies that all the derivatives that can be written on an underlying financial asset are redundant. It makes it possible to hedge any derivative with the underlying. We therefore assume that for the purpose of pricing the Switch options, we are in a "risk-neutral" world, the underlying variable value follows the same process as earlier:

$$\frac{dS_t}{S_t} = (r - \delta)\, dt + \sigma dB_t,\; S_0 = 0$$

Since there is only one source of noise, the market is complete.

Let us now define the following path functionals

$$\begin{aligned} g_t^a(X) &= \sup\{0 \le s \le t : X_s = a\} \\ m(t,T)(X) &= \inf_{u\in[t,T]} X_u \\ T_a(X) &= \inf\{s \ge 0 : X_s = a\}. \end{aligned}$$

Respectively, they are the last time the process crossed level a, the minimum, the maximum, and the hitting time of a. In the following, we will consider two barriers a and d (activating and deactivating), and we will assume that $a \ge d$, since this is the usual situation in the real option approach. The method to derive the results in the opposite case is the same, and in view of our application to real option theory, it does not serve any purpose

A Switch option can start its life being already activated, or deactivated. A particular case of Switch option would be a special "second chance" knock-out call, that could be reactivated any time after it has been knocked out, just by hitting another level. Such an option would have a zero payoff if it is not activated at maturity.

These options are clearly distinct from classical barrier options

- Even if the "inactivated" payoff is zero, they are never worth zero unless at maturity they are not exercised
- The reactivation feature provides the holder with a sort of insurance against a worst-case scenario.
- In a situation of high volatility, a barrier option would have more chances to be cancelled, whereas if it is also true for the Switch option, it has also more chances of being reactivated.

We can consider two cases, that is whether the option starts as being active, or inactive. If the option is inactive, then its payoff will be, for a maturity T

$$p(S_T)\left(\mathbb{I}_{T_a\le T}\mathbb{I}_{m\left(g_T^a,T\right)\ge d}\right) + q(S_T)\left(1 - \mathbb{I}_{T_a\le T}\mathbb{I}_{m\left(g_T^a,T\right)\ge d}\right).$$

This expressions means that if the option starts as inactive, it will pay p if it is activated, and does not hit the deactivating barrier after its latest activation, and q in the other case. The value of an option that starts by being active can be derived from the value of the inactive option. Indeed, the active option will pay: either if it is never deactivated, or if it is deactivated; it will become an option that starts by being inactive at the time it is deactivated.

We can write, thanks to the classical option pricing theory, the price of an inactive option at time 0 and maturity T:

$$V_d(x,T) = e^{-rT}\mathbb{E}_x\left[p(S_T)\left(\mathbb{I}_{T_a\le T}\mathbb{I}_{m\left(g_T^a,T\right)\ge d}\right) + q(S_T)\left(1-\mathbb{I}_{T_a\le T}\mathbb{I}_{m\left(g_T^a,T\right)\ge d}\right)\right].$$

We have the following

Proposition 30 *The price of the inactive switch option is*

$$\begin{aligned}
V_d(x,T) &= \frac{1}{2}e^{-\left(r+\frac{\xi^2}{2}\right)T}\int_0^T \mathbb{P}(T_b\in dt)\int_0^{T-t}\mathbb{P}(g_{T-t}\in dv) \\
&\quad \left\{\int_0^{+\infty} dz f\left(b-\sqrt{T-t-v}z\right)ze^{-\frac{z^2}{2}}\right. \\
&\quad \left. + \int_0^{\frac{b-c}{\sqrt{T-t-v}}} dz f\left(\sqrt{T-t-v}z-b\right)\sum_{k\in\mathbb{Z}}\left(z+2k\frac{b-c}{\sqrt{T-t-v}}\right)e^{-\frac{\left(z+2k\frac{b-c}{\sqrt{T-t-v}}\right)^2}{2}}\right\} \\
&\quad + e^{-\left(r+\frac{\xi^2}{2}\right)T}\int_{-\infty}^{+\infty}\frac{dze^{-\frac{z^2}{2}}}{\sqrt{2\pi}}q\left(xe^{\sigma z}\right)
\end{aligned}$$

with

$$\begin{aligned}
f(y) &= e^{\xi y}\left(p\left(xe^{\sigma y}\right)-q\left(xe^{\sigma y}\right)\right) \qquad (7.3)\\
b &= \frac{1}{\sigma}\ln\left(\frac{a}{x}\right) \\
c &= \frac{1}{\sigma}\ln\left(\frac{d}{x}\right) \\
\xi &= \frac{1}{\sigma}\left(r-\delta-\frac{\sigma^2}{2}\right) = \frac{1}{\sigma}\left(\mu-\frac{\sigma^2}{2}\right)
\end{aligned}$$

We can chose simplified parameters, so as to make the results clearer. We assume $\mu=\frac{\sigma^2}{2}$ so that $\xi=0$. Also, we are interested in a simple payoff function: $p=1$ and $q=0$; then $f=1$. This option pays 1 if it is activated at maturity, and nothing otherwise. Its value in this case is

$$\begin{aligned}
&\frac{1}{2}e^{-rT}\int_0^T \mathbb{P}(T_b\in dt)\int_0^{T-t}\mathbb{P}(g_{T-t}\in dv) \\
&\int_0^{\frac{b-c}{\sqrt{T-t-v}}} dz\sum_{k\in\mathbb{Z}}\left(z+2k\frac{b-c}{\sqrt{T-t-v}}\right)e^{-\frac{\left(z+2k\frac{b-c}{\sqrt{T-t-v}}\right)^2}{2}} \\
=\ & e^{-rT}\int_0^T \mathbb{P}(T_b\in dt)\int_0^{T-t}\mathbb{P}(g_{T-t}\in dv)\sum_{k\in\mathbb{Z}}e^{-2k^2\frac{(b-c)^2}{T-t-v}}\left(1-e^{-\frac{1+4k}{2}\frac{(b-c)^2}{T-t-v}}\right) \\
=\ & e^{-rT}\sum_{k\in\mathbb{Z}}\int_0^T dt\frac{|b|}{\sqrt{2\pi t^3}}e^{-\frac{b^2}{2t}}\int_0^{T-t}\frac{dv}{\pi\sqrt{v}}e^{-2k^2\frac{(b-c)^2}{v}}\left(1-e^{-\frac{1+4k}{2}\frac{(b-c)^2}{v}}\right).
\end{aligned}$$

By comparison, the value of a knock-in option that pays 1 if it has been activated (knocked-in) would be:

$$e^{-rT}\mathbb{P}(T_b\le T) = e^{-rT}\int_0^T dt\frac{|b|}{\sqrt{2\pi t^3}}e^{-\frac{b^2}{2t}},$$

so the difference between the two is

$$e^{-rT}\int_0^T dt\frac{|b|}{\sqrt{2\pi t^3}}e^{-\frac{b^2}{2t}}\left(1-\sum_{k\in\mathbb{Z}}\int_0^{T-t}\frac{dv}{\pi\sqrt{v}}e^{-2k^2\frac{(b-c)^2}{v}}\left(1-e^{-\frac{1+4k}{2}\frac{(b-c)^2}{v}}\right)\right).$$

The proof of the Proposition follows.

Proof. We write $\xi=\frac{1}{\sigma}\left(r-\delta-\frac{\sigma^2}{2}\right)$, and we get

$$\begin{aligned}V_d(x,T) &= e^{-rT}\mathbb{E}\left[\left(p\left(xe^{\sigma B_T^\xi}\right)-q\left(xe^{\sigma B_T^\xi}\right)\right)\mathbb{I}_{T_{\frac{1}{\sigma}\ln\left(\frac{a}{x}\right)}\leq T}\mathbb{I}_{m\left(g_T^{\frac{1}{\sigma}\ln\left(\frac{a}{x}\right)},T\right)\geq\frac{1}{\sigma}\ln\left(\frac{d}{x}\right)}\right]\\ &\quad +e^{-rT}\mathbb{E}\left[q\left(S_T\right)\right].\end{aligned}$$

Thanks to Girsanov's theorem, we have

$$\begin{aligned}&V_d(x,T)\\ =\ & e^{-rT-\frac{\xi^2}{2}T}\mathbb{E}\left[e^{\xi B_T}\left(p\left(xe^{\sigma B_T}\right)-q\left(xe^{\sigma B_T}\right)\right)\mathbb{I}_{T_{\frac{1}{\sigma}\ln\left(\frac{a}{x}\right)}\leq T}\mathbb{I}_{m\left(g_T^{\frac{1}{\sigma}\ln\left(\frac{a}{x}\right)},T\right)\geq\frac{1}{\sigma}\ln\left(\frac{d}{x}\right)}\right]\\ &+e^{-rT-\frac{\xi^2}{2}T}\mathbb{E}\left[e^{\xi B_T}q\left(xe^{\sigma B_T}\right)\right].\end{aligned}$$

We will focus on the first term in the above sum. It is natural then to study an expression of the following form, for a Brownian Motion B,

$$\begin{aligned}\mathbb{E}\left[f(B_T)\mathbb{I}_{T_b(B)\leq T}\quad\mathbb{I}_{m_{g_T^b,T}(B)\geq c}\right] &= \int_0^T\mathbb{P}(T_b\in dt)\,\mathbb{E}\left[f(B_T)\mathbb{I}_{m\left(g_T^b,T\right)\geq c}\middle| T_b=t\right]\\ &= \int_0^T\mathbb{P}(T_b\in dt)\,\mathbb{E}_b\left[f(B_{T-t})\mathbb{I}_{m\left(g_T^b,T\right)\geq c}\right]\end{aligned}$$

thanks to the independence of the Brownian paths $(B_t,t\leq T_b)$ and $(B_{t+T_b}-b,t\geq 0)$. Now we can write

$$\begin{aligned}\mathbb{E}_b\left[f(B_{T-t})\mathbb{I}_{m_{g_{T-t}^b,T-t}\geq c}\right] &= \mathbb{E}\left[f(B_{T-t}+b)\mathbb{I}_{m(g_{T-t},T-t)\geq c-b}\right]\\ &= \int_0^{T-t}\mathbb{P}(g_{T-t}\in dv)\,\mathbb{E}\left[f(B_{T-t}+b)\mathbb{I}_{m(v,T-t)\geq c-b}\middle| g_{T-t}=v\right].\end{aligned}$$

Thus, we have to use the law of the last Brownian excursion away from b before $T-t$. We can write this using the Brownian Meander[4] and conditioning by whether the excursion straddling $T-t$ is above or under 0.

$$\begin{aligned}&\mathbb{E}\left[f(B_{T-t}+b)\mathbb{I}_{m(v,T-t)\geq c-b}\middle| g_{T-t}=v\right]\\ =\ &\frac{1}{2}\mathbb{E}\left[f\left(-\sqrt{T-t-v}m_1+b\right)\right]\\ &+\frac{1}{2}\mathbb{E}\left[f\left(\sqrt{T-t-v}m_1-b\right)\mathbb{I}_{\sup_{u\leq 1}m_u\leq\frac{b-c}{\sqrt{T-t-v}}}\right].\end{aligned}$$

Where m is a Meander (do not confuse with the minimum functional that we have noted $m(a,b)$). We have used the scaling property to simplify the expression. Now, we have

$$\mathbb{E}\left[f\left(-\sqrt{T-t-v}m_1+b\right)\right]=\int_0^{+\infty}dxf\left(b-\sqrt{T-t-v}x\right)xe^{-\frac{x^2}{2}}$$

[4] For a precise definition and some comments, please see Chapter 3.

using the well-known law (cf Revuz and Yor (1991) or Yor (1995)). Now, using heorem 31 (p. 131), we can write

$$\mathbb{E}\left[f(m_1)\mathbb{I}_{\sup_{u\leq 1} m_u \leq y}\right] = \int_0^y dz f(z)(z+2ky)\sum_{k\in\mathbb{Z}} e^{-\frac{(z+2ky)^2}{2}}$$

which gives, after simplifications:

$$\begin{aligned} & \mathbb{E}\left[f(B_T)\mathbb{I}_{T_b(B)\leq T}\mathbb{I}_{m\left(g_T^b,T\right)(B)\geq c}\right] \\ = & \frac{1}{2}\int_0^T \mathbb{P}(T_b\in dt)\int_0^{T-t}\mathbb{P}(g_{T-t}\in dv) \\ & \left\{\int_0^{+\infty} dx f\left(b-\sqrt{T-t-v}x\right)xe^{-\frac{x^2}{2}}\right. \\ & \left. +\int_0^{\frac{b-c}{\sqrt{T-t-v}}} dz f\left(\sqrt{T-t-v}z-b\right)\sum_{k\in\mathbb{Z}}\left(z+2k\frac{b-c}{\sqrt{T-t-v}}\right)e^{-\frac{\left(z+2k\frac{b-c}{\sqrt{T-t-v}}\right)^2}{2}}\right\}. \end{aligned}$$

In this expression, the laws of T_b and g_{T-t} are known. We have

$$\begin{aligned} \mathbb{P}(T_b\in dt) &= dt\frac{|b|}{\sqrt{2\pi t^3}}e^{-\frac{b^2}{2t}} \text{ and} \\ \mathbb{P}(g_{T-t}\in dv) &= \frac{dv}{\pi\sqrt{T-t-v}}. \end{aligned}$$

So we can finally write the result and complete the proof. ■

The price of the active option can also be obtained. As we discussed above, the active option becomes an inactive option when it is deactivated, or just pays at maturity if it is never deactivated. Its price is therefore

$$V_a(x,T) = \mathbb{E}_x\left[e^{-rT_d}V_d(d,T-T_d)\mathbb{I}_{T_d\leq T}\right] + e^{-rT}\mathbb{E}_x\left[p(S_T)\mathbb{I}_{T_d>T}\right].$$

It is very clear in this expression that such a price is decomposed into the price of a classical down and out option $e^{-rT}\mathbb{E}_x\left[p(S_T)\mathbb{I}_{T_d>T}\right]$ (as it pays only if the cancelling barrier is not hit) and the price of this "second renewable chance" $\mathbb{E}_x\left[e^{-rT_d}V_d(d,T-T_d)\mathbb{I}_{T_d\leq T}\right]$, itself matching the value of a Switch option starting deactivated.

Let us write the price of the down and out option as $B^{do}(x,T)$. We obtain the price of the deactivated Switch option by writing the law of the first hitting time, which is well known. Indeed, we have

$$\mathbb{P}(T_d\in dt) = \frac{\frac{1}{\sigma}\ln\left(\frac{S_0}{d}\right)}{\sqrt{2\pi t^3}}\exp\left(-\frac{1}{2t}\left(\frac{1}{\sigma}\ln\left(\frac{S_0}{d}\right)-\xi t\right)\right)$$

and it allows us to write

$$\begin{aligned} V_a(x,T) = & \int_0^T e^{-rt}\frac{\frac{1}{\sigma}\ln\left(\frac{S_0}{d}\right)}{\sqrt{2\pi t^3}}\exp\left(-\frac{1}{2t}\left(\frac{1}{\sigma}\ln\left(\frac{S_0}{d}\right)-\xi t\right)\right)V_d(d,T-t)\,dt \\ & +B^{do}(x,T). \end{aligned}$$

Relationship between Switch options and real options.

Switch options as a replication tool

Let us consider a switch option, the payoff of which at maturity is set to equal the value at that time of an investment project with possible entry and exit. The underlying variable is supposed to be a commodity, traded on a market, so that no-arbitrage arguments are valid for pricing purposes. The barriers of the option are chosen so that they are equal to the thresholds of the investment project in question. Therefore, at maturity, whether the investment is active or not, the switch option replicates the project's value. In other words,

$$\begin{aligned}\text{Value of investment project today} \quad = \quad & \text{Value of Switch option today} \\ & \text{whose payoff equals the value} \\ & \text{of the investment project at time } T.\end{aligned}$$

This means that the switch option also replicates the value of the investment at any time since its inception. Hence, buying the switch option is equivalent, in terms of cashflows, to investing and following the optimal entry/exit decision rule. Equivalently, anyone possessing shares in the investment can perfectly hedge them thanks to the switch option. The switch option therefore also constitutes an option to enter into a project, by providing the holder with exactly the necessary amount of cash at maturity to buy the project. It is a financial option written on a real option.

An investment decision typically generates a continuous stream of cashflows, when it is active, and nothing when an exit decision has just been made. If instead the investor buys a switch option, aiming at "exercizing" it at maturity and then buying into the project, he will pay the premium, and then receive nothing. In fact, it is the appreciation in value of the option which compensates for the missing stream of cashflows.

Switch options to hedge entry and exit costs

In our analysis, we are more interested in Switch options as a way to hedge the future cost of entering or exiting a business, rather than as replication tools. When a project or a firm changes hands, the new management typically will need to implement new strategies, so that they can extract more value from the business. This could be because the firm was badly run, or because of a special know-how that allows them to reduce current costs. In any case, it is fair to assume that the new management, within the framework of entry/exit decisions and real options, will set up different thresholds from the ones in place.

We believe that in most cases, there is a significant delay between the buying of a business (in fact, that is the instant when the buyer decides to buy the business) and the time when the buyer is able to implement his strategy. This delay comes from the time it takes to close the acquisition and restructure the reporting lines in the firm[5].

Switch options provide a hedging vehicle during this delay. Let us write a' and d' the activating and deactivating levels following the new strategy, associated with entry and exit costs of C_e' and C_x' (which may be different from the costs incurred by the previous owners/managers). Note that if the costs are lower for the buyer,

[5] Recent examples, such as the buying of JP Morgan by Chase, or DLJ by CSFB, illutrate the delay can easily be of 6 months.

we would expect $a' \leq a$ and $d' \geq d$. If however the firm was not optimally run, the new optimal barriers could be anywhere with respect to the previous ones.

At time 0, the buyer has decided to acquire the business and possesses a majority of its shares (acquired at the market price, and therefore pricing in the previous strategy). The buyer reckons that he will be able to implement his new strategy only at time T. The buyer incurs the risk that, at time T, the new optimal strategy will require an immediate change in the firm's activity level:

- if $a' \leq a$ then the buyer will need to activate the business and pay C'_e if $a' \leq S_T \leq a$ and the business is not active at time T
- if $a' \geq a$ then the buyer will need to deactivate the business and pay C'_x if $a' \leq S_T \leq a$ and the business is active at time T
- if $d' \geq d$ then the buyer will need to deactivate the business and pay C'_x if $d' \leq S_T \leq d$ and the business is active at time T
- f $d' \leq d$ then the buyer will need to activate the business and pay C'_e if $d' \leq S_T \leq d$ and the business is not active at time T

A simple Switch option allows us to hedge these risks. For example, if $a' \geq a$ and $d' \leq d$, then the buyer would need to be long one Switch option that pays

- C'_e if deactivated with levels a and d and if $d' \leq S_T \leq d$
- C'_x if activated with levels a and d and if $a' \leq S_T \leq a$.

The value of the option at time zero would therefore be given by 7.3 , with

$$\begin{aligned} p(z) &= C'_x \mathbb{I}_{a' \geq z \geq a} \\ q(z) &= C'_e I_{d' \geq z \geq d} \end{aligned}$$

To illustrate this calculation, let us price one leg of this Switch option. We are interested in the part of the option that pays C'_e if the project is not activated at time T and $d' \leq S_T \leq d$. Let us see what happens if the new owner wants to lower the exit threshold significantly. This would make sense if the business was previously run with an excessively high exit level: in this case, exiting often costs a lot, based on the example numbers we showed in the first section. Let us assume that $\mu = \frac{\sigma^2}{2}$ so as to simplify the calculations.

If we use the same parameters as in the numerical examples of the first section, with $a = 120$ and $d = 80$, and $d' = 40$ (so the inactive project has a value of about 96 today), we find that the option is worth about 15% of C'_e. Therefore, if C'_e is the same as C_x (say 500), then the new owner could pay 75 for the Switch option. This option will pay the new owner the entry cost of 500 in a year, if and only if the business is not active while it should be active. Since the value of the project with the new barriers of 120 and 40 is 262, that is 166 over its current cost with the sub-optimal barriers of 120 and 80, the cost of buying the Switch option is well ompensated.

The joint law of the Brownian meander and its running supremum

We have the following:

Theorem 31 *For any measurable positive or bounded function f we have*

$$\mathbb{E}\left[f(m_1)\mathbb{I}_{\sup_{u\leq 1} m_u \leq y}\right] = \int_0^y dz f(z) \sum_{k\in\mathbb{Z}} (z+2ky)\, e^{-\frac{(z+2ky)^2}{2}}$$

where m is a Brownian meander $\left(m_u = \frac{1}{\sqrt{t-g_t}}\left|B_{g_t+u(t-g_t)}\right|, 0 \leq u \leq 1\right)$ for any positive t.

As could be expected, we can check that

$$\begin{aligned}
\lim_{y\to\infty} \mathbb{E}\left[f(m_1)\mathbb{I}_{\sup_{u\leq 1} m_u \leq y}\right] &= \int_0^\infty dz f(z)\, z e^{-\frac{z^2}{2}} = \mathbb{E}\left[f(m_1)\right] \text{ and} \\
\lim_{y\to 0} \mathbb{E}\left[f(m_1)\mathbb{I}_{\sup_{u\leq 1} m_u \leq y}\right] &= 0.
\end{aligned}$$

Proof. The proof relies on Imhof's theorem and on the explicit expression of running supremum densities for Bessel-3 processes. These densities are known and can be directly obtained from Borodin and Salminen. They give the joint law of a Bessel-3 process starting from $x > 0$ and its running maximum (formula 1.1.8, p. 317):

$$\mathbb{P}_x\left(R_t \in dz, \sup_{s\leq t} R_s \leq y\right) = \frac{z}{x\sqrt{2\pi t}} \sum_{k\in\mathbb{Z}} \left(e^{-\frac{(z-x+2ky)^2}{2t}} - e^{-\frac{(z+x+2ky)^2}{2t}}\right) dz.$$

A limit calculation gives the law for a Bessel-3 process starting from 0.

$$\mathbb{P}_0\left(R_1 \in dz, \sup_{s\leq 1} R_s \leq y\right) = \mathbb{I}_{z\leq y} \frac{z\sqrt{2}}{\sqrt{\pi}} \sum_{k\in\mathbb{Z}} (z+2ky)\, e^{-\frac{(z+2ky)^2}{2}} dz.$$

Thanks to Imhof's theorem (see Yor (1997) or Imhof (1984)), we have a relationship between the Brownian meander and a Bessel-3 process. Namely, we have

$$M|_{\mathcal{F}_1} = \left(\sqrt{\frac{\pi}{2}}\frac{1}{X_1}\right) R_0^{(3)}\Big|_{\mathcal{F}_1}$$

where M is the law of the meander between 0 and 1 and R is the law of a Bessel-3 process starting from zero up to 1. Applying this result gives

$$\begin{aligned}
\mathbb{E}\left[f(m_1)\mathbb{I}_{\sup_{u\leq 1} m_u \leq y}\right] &= \mathbb{E}_0\left[f(R_1)\sqrt{\frac{\pi}{2}}\frac{1}{R_1}\mathbb{I}_{\sup_{u\leq 1} R_u \leq y}\right] \\
&= \int_0^y dz f(z) \sum_{k\in\mathbb{Z}} (z+2ky)\, e^{-\frac{(z+2ky)^2}{2}} \\
&\quad \text{which can also be written} \\
&= \sqrt{2\pi} \sum_{k\in\mathbb{Z}} \mathbb{E}\left[f(N-2ky)\, N \mathbb{I}_{(2k+1)y\geq N\geq 2ky}\right] \\
&\quad \text{for } N \text{ a normal Gaussian.}
\end{aligned}$$

This ends the proof. ■

Concluding remarks

We have proposed a new class of barrier derivatives, Switch options, that allows to mitigate the losses due to the "knock-out" effect of classical barrier options. These derivative products also constitute a hedging tool of the business risk linked to entry or exit decision. Switch options can replicate the exit or entry costs that a buyer of a business might have to pay so as to implement his optimal entry and exit strategy.

We required that the underlying business variable should be traded, which would restraint the use of these derivatives mostly to commodities firms, unless that assumption is made that investors are risk-neutral. As a tool to price Switch options, we have derived the joint law of the Brownian Meander and its running maximum.

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Chapter 8 MANAGER'S OPPORTUNISTIC TRADING[1]

Real options allow managers to optimize their firm's value with respect to the economic environment. But there exists an "insider option" for informed agents: the option to arbitrage the market price based on privileged information about the firm's projects. The decision of when to buy or sell and how much is indeed a real option based on market conditions. The exercise of such an option affects market prices.

The impact of transactions on prices, and the motivation for such transactions, have been studied from many perspectives. Jarrow (1994) proposed to analyze the behavior of a manipulator, an agent with a sufficient size to influence the market, who profits from trading on securities and their derivatives at the same time. Back (1992) also focused on the relationship between derivatives prices and their underlyings when some agents can observe the other's order flow. Jeanblanc-Piqué (1992) derived the optimal interest rates manipulation behavior for a central bank. A series of papers, Platen and Schweizer (1994 and 1998), Frey (1996), and Frey and Stremme (1996 and 1998) have more particularly focused on the modeling of the feedback effects of transactions in hedging derivatives. The microstructure literature has also extensively analyzed the relationships between prices and volumes, as explained by O'Hara (1995).

In this chapter we study the optimal arbitrage transactions an informed agent carries out and their influence on the market price. We model the discrepancy between the market price and the real value, known to the informed agent (maybe with some noise), and the impact on the market price of the trading strategy that maximizes the agent's wealth. In our approach, the impact on prices is a consequence of informed trading, and manipulation is not an objective, rather a constraint. The effect of a transaction on the price of a security determines how much it costs to trade this security, as well as the evolution of this price, which conditions future trading gains.

In particular we are interested in the behavior of the manager of an invested perpetual project as an informed trader. The shares of the project, as a subsidiary or as an independent entity, are assumed to be listed on an exchange. The manager can tell the difference between the market price of the project and its real value, due to his privileged information. As Jordan (1998) mentions it:

> [...] I find that managers alter their holdings when the firm's prospects change. Managers consistently take advantage of private firm-specific information, earning positive abnormal returns on open market purchases, while avoiding negative abnormal returns on open market sales.

The decision the manager can make, based on that information, to invest more in that project or to disinvest, that is to buy or to sell shares, corresponds to an exit or entry decision. In this situation, the trades the manager can execute will have an impact on the share price, and they will realign it progressively towards

[1] A SHORTER VERSION OF THIS CHAPTER IS TO APPEAR IN THE *INTERNATIONAL JOURNAL OF APPLIED AND THEORETICAL FINANCE* UNDER THE TITLE "INFORMED OPPORTUNISTIC TRADING AND PRICE OPTIMAL CONTROL".

the real value, which he is the only one to observe. This phenomenon is recognized in the financial community, as could be exemplified by the "Lex Collumn" of the Financial Times (July 6, 1998):

> Actions speak louder than words. That is why monitoring the buying and selling of shares by directors is worth the investor's while.

Specifically, the questions we address are: what is the optimal strategy for the manager? Also, what dynamics will it involve for the share price? And finally, what is the total present value of this information monopole? We would hope this value is under the typical punishment applied to insider dealers. Though, it has been shown by empirical evidence that the current legislation does not prevent insiders from profitably trading shares in their firm. In the 1980's, increased sanctions did not deter most insiders from profitable trading, as many important trials showed it.

In the first section, we present an extension of two models accounting for the influence of transactions on prices, originally designed to model the effect of hedging derivatives. We extend the results of these models to the case of discrete transactions and obtain simple relationships between traded volumes and price jumps. We find that in the case of an infinite market size, the traded volume makes prices jump by an exponential of this volume multiplied by a constant factor. This factor is identified as the depth of the market. For the other model, which defines the market size, the jump depends on the ratio of the number of shares in the market before the trade to the number of shares after the trade. We find that for very small volumes, the infinite market size model is a more tractable proxy for the finite market size.

In the second section, we develop a probabilistic approach to the problem of controlling a Brownian Motion to maintain it in a tunnel. These results have been usually derived using differential equations. Getting an explicit formula for the value function of the stochastic control problem allows us to deal with various constraints.

In the third section, we study the optimal strategy followed by the manager. We propose a simulation that gives the profit the manager can realize by entering these informed trades and influencing the market. We study the optimal range of intervention for the manager, and its implication on the path of the underlying security. We find that the informed traders will maintain the market price in an optimal range around the real value. This affects the distribution of shares performance, and increases its volatility. On the other hand, it ensures that market price are not too far from the real value. Finally, for reasonable parameters, we find the gain (for perpetual trading) to the informed trader can represent about 10% of the company's market capitalization if the correlation between its real and market values is low enough.

The fourth section concludes the paper, followed by the references in a fifth section and an appendix in the sixth section.

A model for the market impact of transactions

In this section we present two different models and study their consequences for the mechanics of the impact of transactions on prices. These models have been designed in the first place to study the continuous hedging of derivatives when transactions influence price. We adapt them to the case of discrete transactions.

Infinite market size model

This model has been introduced in Platen and Schweizer (1994), and was derived from the model presented in Föllmer and Schweizer (1993). The original objective of Föllmer and Schweizer's paper is to study the ergodic behavior of securities prices when time goes to infinity.

We first review the basic assumptions of this model. For each n, we will implicitly define the discrete time price process $\left(S^n_{t_k}, t_k = \frac{T}{n}k\right)$. For each of these n, we consider the case of an economy with A agents, where the authors set the excess demand at time index k of agent a for the underlying asset to be

$$e^n_a(t^n_k, S^n_k) = G^n_a(t^n_{k-1}, \ln \frac{S^n_k}{\widehat{S^n_{a,k}}}) + f^n_a(t^n_{k-1}, \ln S^n_{k-1}, t^n_k, \ln S^n_k) + \lambda^n_{a,k} \tag{8.1}$$

where we note for short $S^n_k = S^n_{t_k}$, and where $\left(\lambda^n_{a,t_k}, t_k = \frac{T}{n}k\right)$ is the only random source in the model. The excess demand is the new demand with respect to what the agents already posses. The first component of this equation, G^n_a, is due to the "arbitrage demand" i.e. the demand describing the reaction of agent a to the deviation of the traded price at date t_k from his assessment $\widehat{S^n_{a,k}}$ of the price for this date. We take $\widehat{S^n_{a,k}} = S^n_{k-1}$. The second component, f^n_a, is the strategy of a market manipulator. The third component, $\lambda^n_{a,k}$, represents the liquidity demand for the asset.

The market clearing condition is

$$\sum_{a \in A} e^n_a(t^n_k, S^n_k) = 0$$

For a given sequence $(t^n_k)_{k \in \{1,\ldots,n\}}$ of trading dates, this yields a sequence $(S^n_k)_k$ of prices which can be viewed as a stochastic process in discrete time.

It is important to clarify the concept of a manipulator as opposed to the rest of the market. We have to consider that if there was no manipulator, the market would follow some dynamic, consistent with past observations. But among the traders we consider as "noise", there could be also manipulators. We consider that, these manipulators being unpredictable from our manipulator's viewpoint, they still remain noise. This explains the importance of having a model which, when free of manipulations, follows an usual dynamic.

We specify in a simple way the manipulator's strategy:

$$f^n_a(t_{k-1}, \ln S^n_{k-1}, t, \ln S^n_k) = \zeta^n(t_k, \ln S^n_k) - \zeta^n(t_{k-1}, \ln S^n_{k-1}).$$

Therefore we assume that the manipulator's strategy depends only on time and on the price of the underlying. We start by ruling out all the strategies which could be based on the history of the underlying's price. In addition to the fact that the set of strategies we consider here is sufficient to encompass a large part of market practices, to allow more complex strategies would result in a purely intractable model. ζ^n represents here the quantity of assets held by the manipulator. Of course, this function does not necessarily depend on n. In Föllmer and Schweizer's original setting, there is no manipulation, as they are interested on the ergodic behavior of the market, depending on the hypotheses made on the shape of the arbitrage demand. Platen and Schweizer, on the contrary, introduce an external influence on the market but confine it to the "technical demand" induced by derivatives hedging.

We assume for simplicity that $(\lambda_k^n)_k$ are i.i.d. square integrable random variables with mean m_n and variance v_n. We now have to specify the form of the arbitrage demand. This demand takes the form

$$\sum_{a \in A} G_a^n(t_{k-1}^n, \ln \frac{S_k^n}{\widehat{S_{a,k}^n}}) = \gamma \ln \frac{S_k^n}{S_{k-1}^n}$$

This assumption implies that the investors make their decisions over a very short horizon in time, which could correspond to short-lived agents. We will suppose throughout the paper that γ is negative, that is investors buy when the price decreases and sell when it increases. This behavior can be considered typical for many investment managers, as their criterium for investment is the discrepancy between the market price of the asset and the fundamental value they assign to it. The reason for this choice is that in the other case, we would be considering a market led only by speculative behaviors. We prefer to model a market where the signal given by price variations is taken as short-term noise, the real meaning behind a lower price being that, for an unchanged fundamental value, the asset is cheaper, thus inducing a willingness to buy.

We then make the assumption that all the functions described in the discrete time setting converge uniformly as n goes to infinity. There are numerous technical conditions to ensure the proper convergence, and they are dealt with in Platen and Schweizer's paper. The authors write the market clearing condition in continuous time

$$0 = \gamma d\left(\ln\left(S_t\right)\right) + d\zeta\left(t, \ln\left(S_t\right)\right) + dU_t \tag{8.2}$$

where U is the limit of the cumulated liquidity demand and writes

$$dU_t = mdt + vdW_t \tag{8.3}$$

with W a Brownian Motion.

The equation 8.2 can be thought of as the simple limit of the discrete time equivalent

$$0 = \gamma \ln \frac{S_k^n}{S_{k-1}^n} + \zeta^n(t_k, \ln S_k^n) - \zeta^n(t_{k-1}, \ln S_{k-1}^n) + \sum_a \lambda_{a,k}^n.$$

The liquidity demand is assumed in fact to converge towards a drifted Brownian Motion.

In the model, U represents the amount held by noisy traders. A part of these traders can be thought of as the impact of the public demand for the financial asset, reflecting the need for cash or the willingness to spare their revenues. Its randomness is supposed to come from the economy. The other part can be thought of as the reactions of institutions as well as consumers to the arrival of information. The above equation is the weak limit of the discrete model, that is the expectation of any functional applied to the discrete market model converges to the same expectation applied to the continuous one. It appears clearly that in this model, the cumulated traded volume over any non-zero interval is infinite, as it equals the variation of a continuous martingale.

We finally have the following

Theorem 32 (Platen Schweizer 1994) *Let $S_0^n = S_0$ for all $n \in \mathbb{N}$. The law of the equilibrium price process $(S_k^n)_{k \in \mathbb{N}}$ converges weakly to the law of the unique* *strong solution* $S = (S_t)_{t\geq 0}$ *of the stochastic differential equation*

$$\frac{dS_t}{S_t} = -\left(\frac{m+\zeta^{\cdot}(t,\ln S_t)}{\gamma+\zeta'(t,\ln S_t)} - \frac{1}{2}\frac{v^2\zeta'(t,\ln S_t)}{\left(\gamma+\zeta'(t,\ln S_t)\right)^2} + \frac{1}{2}\frac{v^2\zeta''(t,\ln S_t)}{\left(\gamma+\zeta'(t,\ln S_t)\right)^3}\right)dt$$
$$-\frac{v}{\gamma+\zeta'(t,\ln S_t)}dW_t \tag{8.4}$$

where $\zeta'(t,\ln S_t)$ *is the derivative of the quantity of assets held by the manipulator expressed in monetary units with respect to* S*, and* $\zeta^{\cdot}(t,\ln S_t)$ *is the derivative with respect to* t *.*

Proof. Refer to the appendix. ■

Platen and Schweizer notice that if there is no manipulation-induced demand, we retrieve the usual geometric Brownian Motion model. In that case, we have

$$\frac{dS_t}{S_t} = -\frac{v}{\gamma}dW_t - \frac{m}{\gamma}dt$$

This particular case allows us to normalize the parameters intervening in the excess demand expression. Indeed, since we want the limit model's constant volatility to equal a given parameter σ, we have $\gamma\sigma = \nu$. In fact, the volatility in the limit model represents the ratio between the liquidity demand's volatility and the intensity of the arbitrage demand. We will give later another interpretation of the parameter γ.

Remarks on the infinite size market model

A precision about volume

As our target is to model the influence of an important transaction on the price dynamic of a financial asset, we start by considering the simplest example of a buying / selling strategy. We suppose the manipulator buys at a constant speed of a asset units per unit of time. In our setting, it corresponds to a manipulation strategy $\zeta(t,\ln S_t) = at$. When the manipulator is buying (or selling, depending on the sign of a), the price dynamic becomes

$$\frac{dS_t^{(a)}}{S_t^{(a)}} = \frac{m+a}{\gamma}dt + \frac{v}{\gamma}dW_t.$$

An issue that can be raised about Platen and Schweizer's model is the infinite volumes: how can a finite volume of transaction, induced by the manipulator, influence the price of the asset whereas the overall traded volume is infinite over any non zero period? A first hint of explanation is that in fact, the observed volume of transaction depends of the scale at which it is observed. Indeed, let us consider an order from a broker: it will be the result of a compensation between all the sell and buy orders of his clients over a certain period of time, which, themselves are the compensations of continuous decisions to buy and sell securities throughout the day based on new information. Thus, the exchanged volume at the market level and at the broker level are not the same, the latter being greater. This aggregating behavior is well documented and comes from issues such as inventory risks, inducing a transaction cost (refer to O'Hara (1995)). We argue that limited transaction volumes are a consequence of these induced costs, and that without transaction costs, volume can be infinite.

Now, instead of the variation of the process as a measure of transaction volume, we propose very simply to measure the difference in the volume held by the traders between two distinct dates, that is, considering the compensation over this period of time. Then, it gives a figure which is comparable with the volume traded by the manipulator. Overall, if we write the price S as the solution of the above stochastic differential equation, it appears clearly that S_t only depends on W_t and not on $(W_u, u \leq t)$, that is, not on the intermediary volume, whatever the precision of its measure. As the transaction volume is induced by the liquidity demand, that is the noisy traders, we can consider that there is an aggregating market-maker at some point.

Finally, let us notice that trading continuously presents an advantage over discrete transactions. Indeed, the noisy agents in our setting cannot see that the price is being manipulated if it is done continuously. A big jump, being a discontinuity, ought to be noticed since these agents do not induce jumps themselves.

A parameter for the depth of the market

Now, let us model the influence of discrete transactions. We start by writing the price when the manipulator is buying at the speed of a. We have

$$S_t^{(a)} = S_0 \exp\left(\left(\frac{m+a}{\gamma} - \frac{v^2}{2\gamma^2}\right)t + \frac{v}{\gamma}W_t\right).$$

One can wonder what one would obtain if discrete transactions were allowed. The following results give the answer.

Proposition 33 *A discrete transaction of size M started at price S_0 makes the price jump to a.s. $S_0 e^{\frac{M}{\gamma}}$. The price paid by the buyer/seller of a discrete block is $MS_0 e^{\frac{M}{\gamma}}$.*

Proof. If the manipulator starts buying at time zero and stops when he holds M units of financial assets, that is at time $\frac{M}{a}$, then we have

$$\begin{aligned}
\mathbb{E}\left[S^{(a)}_{\frac{M}{a}}\right] &= S_0 \mathbb{E}\left[\exp\left(\left(\frac{m+a}{\gamma} - \frac{v^2}{\gamma^2}\right)\frac{M}{a} + \frac{v}{\gamma}W_{\frac{M}{a}}\right)\right] \\
&= S_0 \exp\left(\left(\frac{m+a}{\gamma}\right)\frac{M}{a}\right)
\end{aligned}$$

and therefore

$$\lim_{a\to\infty} \mathbb{E}\left[S^{(a)}_{\frac{M}{a}}\right] = S_0 e^{\frac{M}{\gamma}}.$$

Also,

$$\mathbb{E}\left[\left(S^{(a)}_{\frac{M}{a}} - S_0 e^{\frac{M}{\gamma}}\right)^2\right] = S_0^2 \left(\exp\left(2\left(\frac{m+a}{\gamma} + \frac{v^2}{\gamma^2}\right)\frac{M}{a}\right) - e^{2\frac{M}{\gamma}}\right)$$

and we also have $\lim_{a\to\infty} \mathbb{E}\left[\left(S^{(a)}_{\frac{M}{a}} - S_0 e^{\frac{M}{\gamma}}\right)^2\right] = 0$. This limit results allow us to write that an instant buy order from the manipulator of M units of the asset provokes a jump in the price of almost surely $e^{\frac{M}{\gamma}}$. Therefore, the parameter γ

can be also thought of as a measure of the influence of a direct order on prices: the bigger this number, the smaller the influence. Hence, it can be viewed as the depth of the market.

The expected average price in a continuous transaction can be calculated: it is equal to

$$\frac{a}{M}\int_0^{\frac{M}{a}} dt\mathbb{E}\left[S_t^{(a)}\right] = S_0\frac{a}{M}\int_0^{\frac{M}{a}} dt\exp\left(\left(\frac{m+a}{\gamma}\right)t\right).$$

At the limit, we have $\lim_{a\to\infty}\frac{a}{M}\int_0^{\frac{M}{a}} dt\mathbb{E}\left[S_t^{(a)}\right] = S_0\frac{\gamma}{M}\left(e^{\frac{M}{\gamma}}-1\right)$. However, we will consider that a discrete block trade cannot be traded at the average price. The difference between the terminal price and the average price should be taken up by the intermediaries. The manipulator is a price-taker in this situation, since he/she normally does not have a direct access to the market (not being a broker/dealer). A simple asymptotic analysis tells us that for a small but discrete amount of transaction M, the terminal price is about $\left(1+\frac{M}{\gamma}\right)$ times the starting price, and the average price is about $\left(1+\frac{M}{2\gamma}\right)$ times the starting price.

This completes the proof. ■

The cost of trading and the order queue

We have seen that in this first setting

- A discrete transaction induces a jump in the asset price.
- This price is what the manipulator has to pay when he buys discretely, and it is higher (respectively, lower when he sells) than the listed price an instant before.

It is therefore natural to consider that buying/selling prices constitute bid-ask prices for given quantities.

Proposition 34 *Two discrete immediately consecutive opposed transactions for the same number of units induce a loss of* $S_0\left(e^{\frac{M}{\gamma}}-1\right)$ *per unit.*

Proof. This fact is trivially verified; it is enough to notice that

$$\left(S_0e^{\frac{M}{\gamma}}\right)\gamma\left(1-e^{\frac{-M}{\gamma}}\right) = S_0\gamma\left(e^{\frac{M}{\gamma}}-1\right)$$

and the proof is complete. ■

Let us see how it implies a particular shape of the order queue. We suppose that the quoted price is the last trade price. We also suppose that after every trade, the missing demand or offer which has been "consumed" is immediately replaced in the same proportion as it was before.

The model predicts that for all M, the above relationship should be written

$$\begin{aligned} p &= S_0e^{\frac{M}{\gamma}} \text{ that is} \\ M &= \gamma\ln\left(\frac{p}{S_0}\right). \end{aligned}$$

This explains the link between the volume that can be traded, and the price at which it can be traded: it constitutes the order queue.

The case of a market with a limited number of assets

The market clearing condition in continuous time we saw in the preceding subsection expressed how the random liquidity demand influences the price of an asset. It appears that this influence is not supposed to change with the size of the market, which is the number of assets which can be freely traded. However, if there is a limited total amount of assets in circulation, then the influence of a given transaction should not be the same depending on the size of the market. Indeed, buying 10 million in a 50 million market or in a 50 billion market cannot intuitively have the same effect.

A particular case of the market model which is introduced in Frey and Stremme (1995) accounts for this particular effect. This model, though being also based on the limit of a discrete formulation, does not make the same kind of hypotheses as Platen and Schweizer's one. Frey and Stremme, instead of assuming the existence of two sorts of traders, one being noisy, assume that all the traders' demand is linked to an economic index which, in turn, is randomized. The available wealth of each trader is random and he has to optimize his expected utility with respect to his (simple) trading strategy.

The aggregate demand from the traders for the risky asset, function of the price x, is written $D^n(t_k^n, F_k^n, x)$. Here, F_k^n is a discrete stochastic process which gives the evolution of the economy. This demand function is supposed to verify technical conditions that ensure the existence of the equilibrium, which we do not detail here. To this total demand is added the manipulator's demand, which we write $\xi^n(t_k, x)$. It is different from the manipulation demand mentioned in the preceding model, since here we write the total amount of asset held by the manipulator, and not its variation. The total demand is written

$$G^n(t_k^n, F_k^n, x) = D^n(t_k^n, F_k^n, x) + \xi^n(t_k, x).$$

If we suppose now there are N units of the asset on the market, then the market clearing condition at equilibrium writes (this is the aggregate demand, not excess demand as in Platen and Schweizer's model)

$$G^n(t_k^n, F_k^n, x) = N.$$

Here, Frey and Stremme propose that, given an income f, the trader maximizes

$$D^n\left(F_k^n, x\right) = \arg\max_{d \geq 0} \mathbb{E}\left[u\left(F_k^n + d\left(\widehat{S_{k+1}^n} - x\right)\right)\right]$$

where $\widehat{S_{k+1}^n}$ is the trader's belief of the price in the next period and F_k^n is his income. Frey and Stremme then make the assumptions that

- The utility function is of Constant Relative Risk Aversion, that is $u\left(z\right) = z^{-\gamma}$ for $\gamma > 0$.
- The agent believes that $\widehat{S_{k+1}^n} = x\lambda_k^n$ with $(\lambda_k^n)_k$ independent and identically distributed
- $F_{k+1}^n = F_k^n \epsilon_k^n$ where $(\epsilon_k^n)_k$ independent and identically distributed. If $\epsilon = \lambda$ then we will obtain that the expectations are rational.

Solving the optimization program, the demand is shown to write

$$D^n\left(f, x\right) = \frac{f}{x} D_*^n \tag{8.5}$$

and therefore if there is no manipulation, the market clearing condition, justifying the remark about rational expectations, gives

$$\begin{aligned} S_k^n &= \frac{D_*^n}{N} F_k^n, \text{ that is} \\ S_{k+1}^n &= \frac{D_*^n}{N} F_{k+1}^n = \frac{D_*^n}{N} F_k^n \epsilon_k^n = S_k^n \epsilon_k^n. \end{aligned}$$

Here, the behavior of all the agents is aggregated. In this setting, different agents can have different parameters such as D and F, as long as the sum of all the agents' F equals F_k^n.

Frey and Stremme give the example of an endowment process F converging towards a geometric Brownian Motion. Here, D^n can converge to a constant (or even be a constant), and

$$\epsilon_k^n = \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)\left(t_{k+1}^n - t_k^n\right) + \sigma\sqrt{t_{k+1}^n - t_k^n}\varepsilon_{k+1}^n\right).$$

This is the discretization of a Geometric Brownian Motion and the asset price obviously converges weakly to the solution of

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

where W is a Brownian Motion.

Hence, if there are manipulations, thanks to 8.5, the price solves

$$N = \frac{F_t}{S_t} D + \xi\left(t, S_t\right).$$

Some remarks on the limited size market model

Here, we start by noticing that in the case of the simplest manipulation, we have

$$\begin{aligned} N &= \frac{F_t}{S_t^{(a)}} D + at \\ S_t^{(a)} &= \frac{F_t D}{N - at} = \frac{S_t^{(0)} N}{N - at}. \end{aligned}$$

This process can also be written as

$$S_t^{(a)} = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \ln\left(\frac{N}{N - at}\right) + \sigma W_t\right).$$

Therefore, the influence of buying or selling the asset increases with the number of units already bought or sold. Let us study first the effect of a discrete order of size $M < N$. We have the following

Proposition 35 *A discrete transaction of size $M < N$ provokes a jump in the asset price from S_0 to $S_0 \frac{N}{N-M}$. The price paid by the manipulator is $S_0 \frac{MN}{N-M}$. The transaction cost is $\frac{M}{N-M}$.*

Proof. We follow the proof of the preceding subsection. We write that

$$\mathbb{E}\left[S_{\frac{M}{a}}^{(a)}\right] = \frac{S_0 e^{\mu \frac{M}{a}} N}{N - M}$$

and hence

$$\lim_{a\to\infty}\mathbb{E}\left[S^{(a)}_{\frac{M}{a}}\right]=S_0\frac{N}{N-M}.$$

The convergence of the L^2 norm can also be checked. And this gives the result stated in the proposition. ■

We can compare the results, for the case of a discrete transaction and asymptotically in M, with Platen and Schweizer's model. We see that N and γ play a similar role. Indeed, we notice that

$$S_0\frac{N}{N-M}=S_0\left(1+\frac{M}{N}\right)+o\left(M\right).$$

And for a small amount of transaction M, the transaction cost, as a measure between the price effectively paid and the price before a discrete transaction, is proportional and its coefficient is $\frac{1}{2\gamma}$ under the large market size model. The large market model can therefore be thought of as an approximation of Frey and Stremme's model.

Controlling the BM in a tunnel

We study in this section the optimal control of a Brownian Motion in view of optimizing a functional of the controlled path of the process, the control being performed by keeping the process in a tunnel.

We consider a set of two barriers a and b with $a<b$, and two inner barriers a' and b'. We call $\widehat{B}$ the process derived from B that is sent back to a' (resp. b') whenever it touches a (resp. b).

We are interested in the total value $V\left(x\right)=\mathbb{E}_x\left[\int_0^\infty dte^{-rt}f\left(\widehat{B}_t\right)\right]$ for a negatively bounded function f.

The law of the controlled Brownian Motion

It is natural to consider the problem starting from one of the barriers. We have the following

Proposition 36 *The following equalities hold:*

$$\begin{aligned}V\left(b'\right) &= \int_{a-b'}^{b-b'}dzf\left(z+b'\right)\frac{1}{\sqrt{2r}}\sum_{k\in\mathbb{Z}}\left(e^{-|z+2k(b-a)|\sqrt{2r}}-e^{-|z-2a+2k(b-a)|\sqrt{2r}}\right)\\ &\quad+V\left(b'\right)\frac{\sinh\left(\left(b'-a\right)\sqrt{2r}\right)}{\sinh\left(\left(b-a\right)\sqrt{2r}\right)}+V\left(a'\right)\frac{\sinh\left(\left(b-b'\right)\sqrt{2r}\right)}{\sinh\left(\left(b-a\right)\sqrt{2r}\right)}\end{aligned}$$

and

$$\begin{aligned}V\left(a'\right) &= \int_{a-a'}^{b-a'}dzf\left(z+a'\right)\frac{1}{\sqrt{2r}}\sum_{k\in\mathbb{Z}}\left(e^{-|z+2k(b-a)|\sqrt{2r}}-e^{-|z-2a+2k(b-a)|\sqrt{2r}}\right)\\ &\quad+V\left(a'\right)\frac{\sinh\left(\left(a'-a\right)\sqrt{2r}\right)}{\sinh\left(\left(b-a\right)\sqrt{2r}\right)}+V\left(b'\right)\frac{\sinh\left(\left(b-a'\right)\sqrt{2r}\right)}{\sinh\left(\left(b-a\right)\sqrt{2r}\right)}.\end{aligned}$$

Before going to the proof, notice that the knowledge of

$$F\left(r,f\right)=\mathbb{E}_x\left[\int_0^\infty dte^{-rt}f\left(\widehat{B}_t\right)\right]$$

gives the law of $\widehat{B_t}$. Indeed, inverting the Laplace transform in time would give $\mathbb{E}_x f\left(\widehat{B_t}\right)$ for all f, which in turn allows the characterization of the law of $\widehat{B_t}$.

Proof. We know that, thanks to the strong Markov property of the Brownian Motion,

$$\begin{aligned} V\left(b'\right) &= \mathbb{E}_{b'}\left[\left(\int_0^{T_b} dte^{-rt} f\left(B_t\right) + e^{-rT_b} V\left(b'\right)\right) \mathbb{I}_{T_b \leq T_a}\right] \\ &\quad + \mathbb{E}_{b'}\left[\left(\int_0^{T_a} dte^{-rt} f\left(B_t\right) + e^{-rT_a} V\left(a'\right)\right) \mathbb{I}_{T_a \leq T_b}\right]. \end{aligned}$$

and we want to calculate a quantity of the kind

$$\begin{aligned} &\mathbb{E}_0\left[\left(\int_0^{T_b} dte^{-rt} f\left(B_t\right) + e^{-rT_b} V\left(b'\right)\right) \mathbb{I}_{T_b \leq T_a}\right] \\ =& \int_0^{\infty} dte^{-rt} \mathbb{E}\left[f\left(B_t\right) \mathbb{I}_{t \leq T_b \leq T_a} \mathbb{I}_{\sup_{s \leq t} B_s \leq b}\right] + \mathbb{E}_0\left[e^{-rT_b} V\left(b'\right) \mathbb{I}_{T_b \leq T_a}\right] \end{aligned}$$

But it can also be written

$$\int_0^{\infty} dte^{-rt} \mathbb{E}\left[f\left(B_t\right) \mathbb{I}_{t \leq T_b} \mathbb{I}_{\inf_{u \leq T_b} B_u \geq a} \mathbb{I}_{\sup_{s \leq t} B_s \leq b}\right] + \mathbb{E}_0\left[e^{-rT_b} V\left(b'\right) \mathbb{I}_{T_b \leq T_a}\right]$$

and the integrand equals

$$\mathbb{E}\left[f\left(B_t\right) \mathbb{I}_{\inf_{u \leq t} B_u \geq a} \mathbb{I}_{\inf_{u \in [t, T_b]} B_u \geq a} \mathbb{I}_{\sup_{s \leq t} B_s \leq b}\right].$$

where it is clear that all the characteristic functions are equivalent to that of the set where $t \leq T_b \leq T_a$. By conditioning we get that it equals

$$\begin{aligned} & \mathbb{E}\left[f\left(B_t\right) \mathbb{I}_{\inf_{u \leq t} B_u \geq a} \quad \mathbb{I}_{\sup_{s \leq t} B_s \leq b} \mathbb{E}_{\mathcal{F}_t}\left[\mathbb{I}_{\inf_{u \in [t, T_b]} B_u \geq a}\right]\right] \\ =& \mathbb{E}\left[f\left(B_t\right) \mathbb{I}_{\inf_{u \leq t} B_u \geq a} \mathbb{I}_{\sup_{s \leq t} B_s \leq b} \mathbb{E}_{B_t}\left[\mathbb{I}_{\inf_{u \in [t, T_b]} B_u \geq a}\right]\right] \end{aligned}$$

by the Markov property of the Brownian Motion. Let us now mention the well known formula, that can be found for example in Borodin and Salminen (1996, formula 2.2.2, p. 163)

$$\mathbb{P}_x\left(\inf_{u \leq T_b} B_u \geq a\right) = \frac{x-a}{b-a} \mathbb{I}_{a \leq x \leq b} + \mathbb{I}_{a \leq b \leq x}$$

so that we get

$$\begin{aligned} & \mathbb{E}\left[\left(\int_0^{T_b} dte^{-rt} f\left(B_t\right) + e^{-rT_b} V\left(b'\right)\right) \mathbb{I}_{T_b \leq T_a}\right] \\ =& \int_0^{\infty} dte^{-rt} \mathbb{E}\left[f\left(B_t\right) \frac{B_t - a}{b-a} \mathbb{I}_{\inf_{u \leq t} B_u \geq a} \mathbb{I}_{\sup_{s \leq t} B_s \leq b}\right] + V\left(b'\right) \mathbb{E}\left[e^{-rT_b} \mathbb{I}_{T_b \leq T_a}\right]. \end{aligned}$$

But the joint law of the Brownian Motion and its first exit time of a tunnel $\mathbb{P}\left(B_t \in dy, T_a \wedge T_b > t\right)$ is known, as mentioned in Karatzas and Shreve (1991, Proposition 8.10, P. 90) and we have therefore

$$\begin{aligned} & \mathbb{E}\left[f\left(B_t\right) \frac{B_t - a}{b-a} \mathbb{I}_{\inf_{u \leq t} B_u \geq a} \mathbb{I}_{\sup_{s \leq t} B_s \leq b}\right] \\ =& \int_a^b f(z) \frac{z-a}{b-a} \frac{dz}{\sqrt{2\pi t}} \sum_{k \in \mathbb{Z}} \left(e^{-\frac{(z+2k(b-a))^2}{2t}} - e^{-\frac{(z-2a+2k(b-a))^2}{2t}}\right). \end{aligned}$$

Now, considering a process starting from b' does not change much in the above equation. We get

$$\begin{aligned}
& \mathbb{E}_{b'}\left[f\left(B_t\right)\frac{B_t-a}{b-a}\mathbb{I}_{\inf_{u\le t}B_u\ge a}\mathbb{I}_{\sup_{s\le t}B_s\le b}\right] \\
= & \ \mathbb{E}\left[f\left(B_t+b'\right)\frac{B_t-a+b'}{b-a}\mathbb{I}_{\inf_{u\le t}B_u+b'\ge a}\mathbb{I}_{\sup_{s\le t}B_s+b'\le b}\right] \\
= & \int_{a-b'}^{b-b'} f\left(z+b'\right)\frac{z+b'-a}{b-a}\frac{dz}{\sqrt{2\pi t}}\sum_{k\in\mathbb{Z}}\left(e^{-\frac{(z+2k(b-a))^2}{2t}}-e^{-\frac{\left(z-2\left(a-b'\right)+2k(b-a)\right)^2}{2t}}\right).
\end{aligned}$$

Integrating with respect to t we get the following Laplace transform

$$\begin{aligned}
& \int_0^\infty dt e^{-rt}\mathbb{E}_{b'}\left[f\left(B_t\right)\frac{B_t-a}{b-a}\mathbb{I}_{\inf_{u\le t}B_u\ge a}\mathbb{I}_{\sup_{s\le t}B_s\le b}\right] \\
= & \int_{a-b'}^{b-b'} f\left(z+b'\right)\frac{z+b'-a}{b-a}\frac{1}{\sqrt{2r}}\sum_{k\in\mathbb{Z}}\left(e^{-|z+2k(b-a)|\sqrt{2r}}-e^{-|z-2(a-b')+2k(b-a)|\sqrt{2r}}\right).
\end{aligned}$$

It is also known (refer to Revuz and Yor (1991) for example, or Karatzas and Schreve (1991), p. 100) that

$$\mathbb{E}_{b'}\left[e^{-rT_b}\mathbb{I}_{T_b\le T_a}\right]=\frac{\sinh\left(\left(b'-a\right)\sqrt{2r}\right)}{\sinh\left(\left(b-a\right)\sqrt{2r}\right)}.$$

Now, summing the various expressions we obtain the first equation in the theorem. The second equation is written very easily by changing the variables, and using the fact that $B=-B$ in law. In particular, one has to use that

$$\mathbb{P}_x\left(\sup_{u\le T_a}B_u\ge a\right)=\frac{b-x}{b-a}$$

from Borodin and Salminen (1996, 2.1.2, p. 163) and the proof of the proposition is complete. ■

If we write the density

$$K\left(z\right)=\mathbb{I}_{b-b'\ge z\ge a-b'}\frac{z+b'-a}{b-a}\frac{dz}{\sqrt{2\pi t}}\sum_{k\in\mathbb{Z}}\left(e^{-\frac{(z+2k(b-a))^2}{2t}}-e^{-\frac{(z-2a+2k(b-a))^2}{2t}}\right)$$

then we get by solving the system that

$$V\left(b'\right)=\frac{\alpha+\beta\gamma}{1-\beta\delta}$$

with

$$\begin{aligned}
\alpha &= \frac{\int dz f\left(z+b'\right)K\left(z\right)\sinh\left(\left(b-a\right)\sqrt{2r}\right)}{\sinh\left(\left(b-a\right)\sqrt{2r}\right)-\sinh\left(\left(b'-a\right)\sqrt{2r}\right)} \\
\beta &= \frac{\sinh\left(\left(b-b'\right)\sqrt{2r}\right)}{\sinh\left(\left(b-a\right)\sqrt{2r}\right)-\sinh\left(\left(b'-a\right)\sqrt{2r}\right)} \\
\gamma &= \frac{\int dz f\left(z+a'\right)K\left(z\right)\sinh\left(\left(b-a\right)\sqrt{2r}\right)}{\sinh\left(\left(b-a\right)\sqrt{2r}\right)-\sinh\left(\left(b-a'\right)\sqrt{2r}\right)} \\
\delta &= \frac{\sinh\left(\left(a'-a\right)\sqrt{2r}\right)}{\sinh\left(\left(b-a\right)\sqrt{2r}\right)-\sinh\left(\left(b-a'\right)\sqrt{2r}\right)}.
\end{aligned}$$

Distribution of the cost of control

Now, let us look at the "cost" of controlling the Brownian Motion in a tunnel. We define $C_{a'}$ and $C_{b'}$ as the finite costs of pushing the Brownian Motion into the tunnel up and down, respectively.

Proposition 37 *We have the following result*

$$\begin{aligned} C\left(b'\right) &= \mathbb{E}_{b'}\left[\sum_i \left(e^{-rT_{a',i}}C_{a'} + e^{-rT_{b',i}}C_{b'}\right)\right] \\ &= \frac{C_{b'}\left(\alpha' + \beta\delta\right) + C_{a'}\left(\beta + \gamma'\beta\right)}{1-\beta\delta} \end{aligned}$$

with

$$\begin{aligned} \alpha' &= \frac{\sinh\left(\left(b-a'\right)\sqrt{2r}\right)}{\sinh\left(\left(b-a\right)\sqrt{2r}\right) - \sinh\left(\left(b'-a\right)\sqrt{2r}\right)} \text{ and} \\ \gamma' &= \frac{\sinh\left(\left(b'-a\right)\sqrt{2r}\right)}{\sinh\left(\left(b-a\right)\sqrt{2r}\right) - \sinh\left(\left(b'-a\right)\sqrt{2r}\right)}. \end{aligned}$$

Proof. Starting from b' we obtain easily, relying on the preceding computations:

$$\begin{aligned} C\left(b'\right) &= \mathbb{E}_{b'}\left[e^{-rT_b}\left(C_{b'} + C\left(b'\right)\right)\mathbb{I}_{T_b\le T_a} + e^{-rT_a}\left(C_{a'} + C\left(a'\right)\right)\mathbb{I}_{T_a\le T_b}\right] \\ &= \left(C_{b'} + C\left(b'\right)\right)\frac{sh\left(\left(b'-a\right)\sqrt{2r}\right)}{sh\left(\left(b-a\right)\sqrt{2r}\right)} + \left(C_{a'} + C\left(a'\right)\right)\frac{sh\left(\left(b-b'\right)\sqrt{2r}\right)}{sh\left(\left(b-a\right)\sqrt{2r}\right)}. \end{aligned}$$

And

$$C\left(a'\right) = \left(C_{b'} + C\left(b'\right)\right)\frac{sh\left(\left(a'-a\right)\sqrt{2r}\right)}{sh\left(\left(b-a\right)\sqrt{2r}\right)} + \left(C_{a'} + C\left(a'\right)\right)\frac{sh\left(\left(b-a'\right)\sqrt{2r}\right)}{sh\left(\left(b-a\right)\sqrt{2r}\right)}.$$

So we finally get

$$C\left(b'\right) = \frac{C_{b'}\left(\alpha' + \beta\delta\right) + C_{a'}\left(\beta + \gamma'\beta\right)}{1-\beta\delta}$$

The symmetric expression follows easily. ■

So as to compute the value or the cost of a control strategy for the Brownian Motion starting from any point in the tunnel, one just has to calculate

$$C\left(x\right) = \mathbb{E}_x\left[e^{-rT_b}C\left(b'\right)\mathbb{I}_{T_b\le T_a} + e^{-rT_a}C\left(a'\right)\mathbb{I}_{T_b>T_a}\right]$$

and thanks to the result mentioned above, we have

$$C\left(x\right) = C\left(b'\right)\frac{\sinh\left(\left(x-a\right)\sqrt{2r}\right)}{\sinh\left(\left(b-a\right)\sqrt{2r}\right)} + C\left(a'\right)\frac{\sinh\left(\left(b-x\right)\sqrt{2r}\right)}{\sinh\left(\left(b-a\right)\sqrt{2r}\right)}.$$

An optimal control in a very general setting can be found by minimizing the costs while maximizing a functional of the path of the controlled process, over the strategies represented by the set of four barriers.

We now consider a setting where the relevant variable to the investment project or the company, observed by the manager only, follows a geometric Brownian Motion. This variable can be a demand level, a commodity price, or directly the stream of cash-flows generated by the company. The market value of the company is supposed to follow a correlated Brownian Motion, with the same drift. Indeed, we assume the market information is noisy but unbiased. An extension of the model to the biased case is straightforward but computationally heavy. The informed agent knows the real value of the company, which is only correlated to its market value. When the discrepancy between the two reaches an optimal level, the agent will enter trades so as to benefit from the difference in valuation: the cashflows the stock will generate in the future have a certain present value, and its difference with the market value will be a net gain for the agent. Though, while performing these transactions, the agent will influence the price of the stock, and push it towards its real value. The problem is similar to that of controlling a random process to stay in a tunnel.

An investment decision is usually assumed to be perpetual, the investment cost being a sunk cost. The agents we study are not infinitely lived as corporations can be, but we can consider that the impact of a manager's trades should not be too different from his successor, and therefore we can suppose the successive informed traders can be aggregated into one. Some investments can be exited, even if the cost is high. Dixit and Pindyck (1994) have exposed many examples, as well as the methods to address the problem of valuation in that case. It is sometimes possible, indeed, to sell a plant or an expensive machine on the market, and therefore exit the market. But generally, the costs of exiting an investment are important, due for example to the social costs of firing employees. So the optimal investment strategy is to invest when the relevant variable hits an optimal high level, and disinvest each time it hits an optimal low level. In fact, the decision to invest in real option theory correspond to the decision to buy in our problem setting, and disinvesting corresponds to selling. But our problem is even simpler as the only relevant cashflows will only intervene at these times when transactions are performed, instead of being a continuous stream.

The present section will use the mathematical results shown in the preceding section to express closed-form solutions for the price control problem of the informed trader.

The setting of the model

We assume there exists an underlying random process that conditions the company's business (its market share, the price of inputs, the price of the goods sold, ...). We suppose that this variable is governed by the following stochastic differential equation

$$dX_t = X_t \mu dt + X_t \sigma dB_t$$

and the perception of this variable by the market follows

$$dS_t = S_t \mu dt + S_t \sigma dW_t$$

with $\langle B, W \rangle_t = \rho t$.

We write, for the informed manager, the value of the company at time t as

$$V(x) = \mathbb{E}_{X_t = x} \left[\int_t^{\infty} du e^{-ru} f(X_u) \right].$$

In the relatively general case of a Cobb-Douglass profit function for f, that is an exponent function, we obtain something of the form

$$V(X_t) = \Delta X_t^{\kappa}.$$

If f is the identity, we obtain for example

$$V(X_0) = \frac{X_0}{r-\mu}$$

The value to the market investors, that is the market price, will write ΔS_t^{κ}. It is natural to assume that the preferences of the manager and the investors are the same, since if they were not, then there would be a necessary minimal amount of transactions between them. However, if X equalled S then there should be no reason why the investor and the manager would trade in our setting.

When the informed manager enters a trade that allows him to buy shares of the company for less than what they are really worth, or the contrary, his profit has to be considered in terms of yield rather than absolute value. Indeed, we do not put constraints on his ability to get financing or to sell short. Therefore, the right benchmark would be what proportional return he would obtain if he did not trade. Then, by buying more, he increases his yield with respect to the yield of the company (at which he would have invested), and by selling at the right time, he invests the money he's getting at the risk-free rate. The difference in price between the real value and the trade price times the number of shares represents the dollar amount that he saves or gains at each trade, and is comparable to the value of the firm at any time. Taking the present value gives the value today of performing these trades.

The manager looks at the ratio between the real value and the market value, and depending on how far they are from each other, will decide to buy, sell, or do nothing. This is a very simple control problem. Let us write the value of trading $V\left(\frac{s}{x}\right)$ as a function of the ratio. The manager will maximize this value with respect to his parameters, i.e. the levels at which he trades, and the amounts he trades.

In addition, we will consider that the transactions undertaken by the manager must be of a minimum size, and/or maximum size. Indeed, any size cannot be traded at once: there are blocks, representing minimal or maximal sizes. This volume constraint could appear to be an "information asymmetry" cost, as the broker does not know the manager is informed, but knows that some trades are informed, and observing the size of the trades gives some information. O'Hara (1995) for example explains this situation. The minimal or maximal volume at each trade will be noted m.

The value of a strategy in the infinite market model

As soon as the ratio between the value of the company to the manager and to the market goes up to a given level or goes down to a lower threshold, the manager will decide to trade.

The process $Z = \frac{V(S)}{V(X)}$ is a geometric Brownian Motion, and its dynamic is easy to derive. Indeed, we have

$$Z_t = \frac{S_0^{\kappa} e^{\kappa\left(\mu - \frac{\sigma^2}{2}\right)t + \kappa\sigma W_t}}{X_0^{\kappa} e^{\kappa\left(\mu - \frac{\sigma^2}{2}\right)t + \kappa\sigma B_t}} = \frac{S_0^{\kappa}}{X_0^{\kappa}} e^{\kappa\sigma(W_t - B_t)}.$$

So, thanks to the assumption that the drift and the volatility are the same for the market's perception and reality, the expression of Z is greatly simplified. As the process in the exponential has no drift, there won't be a need to use Girsanov's theorem in the calculations. The ratio $\frac{S_0^\kappa}{X_0^\kappa}$ can be considered to be 1 if at the starting time, the market has a perfect information no the firm's value. But if we consider the decision to trade by the manager after the company has been set up, this ratio may be different from 1. W and B being two correlated Brownian Motions, $W-B$ is a Brownian Motion with a multiplicative coefficient of $\sqrt{2(1-\rho)}$.

$$Z_t = \frac{S_0^\kappa}{X_0^\kappa} e^{\kappa\sigma\sqrt{2(1-\rho)}U_t}$$

where U is a standard Brownian Motion.

When Z leaves a tunnel, the manager trades a volume of M (positive in the case of buying and negative if selling). This costs (at time t) $\Delta S_t M J(M)$ where J is the "jump function", and the price jumps by $J(M)$. We can write h_u and h_d the barriers that trigger the trades, and the manager wants the ratio to reach its optimal level l_u or l_d. It implies that there is a precise volume that has to be traded so that the share price adjustment moves the ratio by the desired amount. Propositions 12 and 13 allow us to write explicitly the cost and impact of transactions.

Let us write M_u and M_d the volumes traded respectively in the case of selling or buying. In the case of selling, we have $e^{\frac{M_u}{\phi}} Z_t = l_u$ just after the transaction, where ϕ is the depth of the market. But $Z_t = h_u$, and it implies $M_u = \phi \ln\left(\frac{h_u}{l_u}\right)$. Notice that $Z_t = h_u$ is equivalent to $U_t = \frac{\ln\left(\frac{X_0^\kappa}{S_0^\kappa} h_u\right)}{\kappa\sigma\sqrt{2(1-\rho)}}$, so the hitting time we are interested in would be $T_b(U)$ with $b = \frac{\ln\left(\frac{X_0^\kappa}{S_0^\kappa} h_u\right)}{\kappa\sigma\sqrt{2(1-\rho)}}$.

In the other case, we obtain $e^{\frac{M_d}{\phi}} Z_t = l_d$ and $M_d = \phi \ln\left(\frac{l_d}{h_d}\right)$, which is naturally positive. Notice that simplifying the expressions with $S_0 = X_0$ gives for all the levels

$$\begin{aligned}
b &= \frac{\ln(h_u)}{\kappa\sigma\sqrt{2(1-\rho)}}, \; b' = \frac{\ln(l_u)}{\kappa\sigma\sqrt{2(1-\rho)}}, \\
a' &= \frac{\ln(l_d)}{\kappa\sigma\sqrt{2(1-\rho)}}, \; a = \frac{\ln(h_d)}{\kappa\sigma\sqrt{2(1-\rho)}}.
\end{aligned}$$

The impact of a transaction on the wealth of the manager is simply the number of shares traded times the gain per share, all being divided by the real value of the company (since we are interested in the return generated by informed trading with respect to the value of a share, as a yield). The gain per share equals the real price minus the market price or the opposite, depending if it is a buy or a sell. So we have in the case of buying

$$\begin{aligned}
\text{gain} &= G_d = M_d \frac{\Delta\left(X_t^\kappa - S_t^\kappa e^{\frac{M_d}{\phi}}\right)}{\Delta X_t^\kappa} \\
&= \phi \ln\left(\frac{l_d}{h_d}\right)(1-l_d)
\end{aligned}$$

When selling

$$\text{gain} = G_u = \phi \ln \left(\frac{h_u}{l_u} \right) (h_u - 1) .$$

The additional yield to the manager due to informed trading equals the sum of the present value of the yield of future trades. The additional yield, starting after a sale, writes

$$\mathbb{E}_{b'} \left[\sum_i \left(e^{-rT_d^i} G_d + e^{-rT_u^i} G_u \right) \right]$$

where the times refer to the hitting times of Z. This amount gives, in stock value units, how much the trader can generate by using his privileged information.

So, using proposition 16, we get

$$VT\left(b'\right) = \frac{G_u \left(\alpha' + \beta\delta\right) + G_d \left(\beta + \gamma'\beta\right)}{1 - \beta\delta}.$$

Thanks to the calculations performed in the preceding section, we can therefore write the following

Taking into consideration the volume constraint implies that the intervention boundaries h and l must verify $m = \phi \ln \left(\frac{l_d}{h_d} \right)$ and $m = \phi \ln \left(\frac{h_u}{l_u} \right)$ (or $\leq$ or $\geq$ depending whether the constraint is a fixed volume, a volume greater or equal to m or lesser or equal to m).

Proposition 38 *The value of trading for the informed manager, for given intervention levels h_u and h_d, and given target levels l_u and l_d, under the assumptions pertaining to the extension of Platen and Schweizer's model, we have*

- *When buying, the volume of stock traded writes $M_d = \phi \ln \left(\frac{l_d}{h_d} \right)$ and the additional yield realized writes $G_d = \phi \ln \left(\frac{l_d}{h_d} \right) (1 - l_d)$,*
- *When selling, the volume of stock traded writes $M_u = \phi \ln \left(\frac{h_u}{l_u} \right)$ and the additional yield writes $G_u = \phi \ln \left(\frac{h_u}{l_u} \right) (l_u - 1)$,*
- *The present value to the manager of trading perpetually is*

$$\begin{aligned} V(z) &= V(l_u) \frac{\sinh \left(\ln \left(\left(\frac{z}{h_d} \right)^{\frac{\sqrt{r}}{\kappa\sigma\sqrt{1-\rho}}} \right) \right)}{\sinh \left(\ln \left(\left(\frac{h_u}{h_d} \right)^{\frac{\sqrt{r}}{\kappa\sigma\sqrt{1-\rho}}} \right) \right)} \\ &\quad + V(l_d) \frac{\sinh \left(\ln \left(\left(\frac{h_u}{z} \right)^{\frac{\sqrt{r}}{\kappa\sigma\sqrt{1-\rho}}} \right) \right)}{\sinh \left(\ln \left(\left(\frac{h_u}{h_d} \right)^{\frac{\sqrt{r}}{\kappa\sigma\sqrt{1-\rho}}} \right) \right)} \end{aligned}$$

with a, b, a', b' defined as above and

$$\begin{aligned} V\left(b'\right) &= \frac{G_u \left(\alpha' + \beta\delta\right) + G_d \left(\beta + \gamma'\beta\right)}{1 - \beta\delta} \\ V\left(a'\right) &= \frac{G_u \left(\delta\alpha' + \delta\right) + G_d \left(\gamma' + \delta\beta\right)}{1 - \beta\delta}. \end{aligned}$$

- *If the volume of each trade has to be equal to m, then one has the additional relationships $l_u = h_u e^{-\frac{m}{\phi}}$ and $h_d = l_d e^{-\frac{m}{\phi}}$.*

The value of the strategy in the finite-size market

In this framework, we take into account the outstanding volume of stock on the market. In this model, the variable that impacts prices is not the volume traded at a precise instant, but rather the total volume held by the manipulator. The market price is expressed as a function of this volume, so that variations of this volume have an influence on the price. We can define the "zero manipulation price" as the price's dynamic if there was no manipulation. It is consistent with the model's hypotheses to assume that the zero manipulation price is the price process with the same dynamic as the real price. So we have the following definition for the zero manipulation price

$$\widetilde{S}_t = \widetilde{S}_0^\kappa e^{\kappa\left(\mu-\frac{\sigma^2}{2}\right)t+\kappa\sigma W_t}$$

and using proposition 35, we can write the observed market price as

$$S_t = \widetilde{S}_t \frac{N}{N - M_t}$$

where N is the outstanding volume and M is the quantity held by the manager. The ratio of the real value to the market value therefore writes

$$\frac{N}{N-M_t} \frac{\widetilde{S}_0^\kappa e^{\kappa\left(\mu-\frac{\sigma^2}{2}\right)t+\kappa\sigma W_t}}{X_0^\kappa e^{\kappa\left(\mu-\frac{\sigma^2}{2}\right)t+\kappa\sigma B_t}} = \frac{N}{N-M_t} \frac{\widetilde{S}_0^\kappa}{X_0^\kappa} e^{\kappa\sigma\sqrt{2}\sqrt{1-\rho}U_t}.$$

The relationship between the traded volume and its impact on price depends on the volume that is already held by the manager. Therefore, the optimal barriers will depend on the volume held by the manager (which will evolve through time, depending on the trades undertaken).

If we write $\Delta M_t = M_t - M_{t-}$ the volume traded at time t (an intervention time), then we get in the case of a sale

$$\begin{aligned} Z_{t-} &= h_u = \frac{\widetilde{S}_{t-}}{X_{t-}^\kappa} \frac{N}{N-M_{t-}} \\ Z_t &= l_u = \frac{\widetilde{S}_{t-}}{X_{t-}^\kappa} \frac{N}{N-M_{t-}-\Delta M_t} = h_u \frac{N-M_{t-}}{N-M_{t-}-\Delta M_t} \end{aligned}$$

where the optimal barriers h and l are to be understood as dependent on M. We obtain

$$\Delta M_t = -\frac{(N-M_{t-})(h_u - l_u)}{l_u}.$$

The realized gain, relative to the real value, writes therefore

$$(h_u - l_u)\left|\Delta M_t\right| = \frac{(N-M_{t-})(h_u-l_u)(l_u-1)}{l_u}.$$

minus any transaction costs. Similarly in the case of a buy transaction, one gets

$$\Delta M_t = \frac{(N-M_{t-})(h_d - l_d)}{l_d}$$

and the gain equals

$$(l_d - h_d)\,|\Delta M_t| = \frac{(N - M_{t-})\,(h_d - l_d)\,(1 - l_d)}{l_d}.$$

So under the assumptions of this model, the series of gains realized by the manager depends on his original share in the capital. For a volume M_0 held by the manager at the start, depending whether the first trade is a buy or a sale, the volume he controls will be $M_0 + \frac{(N-M_{t-})(h_d-l_d)}{l_d}$ or $M_0 - \frac{(N-M_{t-})(h_u-l_u)}{l_u}$.

It is possible to write the functional equation that is solved by the optimal intervention levels. Indeed, if we write $V_u\,(M)$ and $V_d\,(M)$ the value of the strategy, starting just after an intervention time (up or down), and if the starting volume is M, we have

$$\begin{aligned}
V_u\,(M) \quad = \quad & \sup_{h_u,l_u,h_d,l_d} \Bigg\{ \mathbb{E}_{l_u} \left[\mathbb{I}_{T_{h_u} \leq T_{hd}} e^{-rT_{h_u}} \frac{(N - M_{t-})\,(h_u - l_u)\,(l_u - 1)}{l_u} \right] \\
& + \mathbb{E}_{l_u} \left[\mathbb{I}_{T_{h_u} \leq T_{h_d}} e^{-rT_{h_u}} V_u \left(M - \frac{(N - M)\,(h_u - l_u)}{l_u} \right) \right] \\
& + \mathbb{E}_{l_u} \left[\mathbb{I}_{T_{h_d} \leq T_{h_u}} e^{-rT_{h_d}} \frac{(N - M_{t-})\,(h_d - l_d)\,(1 - l_d)}{l_d} \right] \\
& + \mathbb{E}_{l_u} \left[\mathbb{I}_{T_{h_d} \leq T_{h_u}} e^{-rT_{h_d}} V_d \left(M + \frac{(N - M)\,(h_d - l_d)}{l_d} \right) \right] \Bigg\}
\end{aligned}$$

and

$$\begin{aligned}
V_d\,(M) \quad = \quad & \sup_{h_u,l_u,h_d,l_d} \Bigg\{ \mathbb{E}_{l_d} \left[\mathbb{I}_{T_{h_u} \leq T_{hd}} e^{-rT_{h_u}} \frac{(N - M_{t-})\,(h_u - l_u)\,(l_u - 1)}{l_u} \right] \\
& + \mathbb{E}_{l_d} \left[\mathbb{I}_{T_{h_u} \leq T_{h_d}} e^{-rT_{h_u}} V_u \left(M - \frac{(N - M)\,(h_u - l_u)}{l_u} \right) \right] \\
& + \mathbb{E}_{l_d} \left[\mathbb{I}_{T_{h_d} \leq T_{h_u}} e^{-rT_{h_d}} \frac{(N - M_{t-})\,(h_d - l_d)\,(1 - l_d)}{l_d} \right] \\
& + \mathbb{E}_{l_d} \left[\mathbb{I}_{T_{h_d} \leq T_{h_u}} e^{-rT_{h_d}} V_d \left(M + \frac{(N - M)\,(h_d - l_d)}{l_d} \right) \right] \Bigg\}
\end{aligned}$$

The barriers values that maximize these expressions give the optimal strategies. The expression can be written explicitly, but the maximization cannot be performed analytically, the only possible approach being then numerical.

Imperfect information

Considering that the informed trader has perfect information can raise a problem. Indeed, there is always some level of uncertainty. It would therefore make sense to assume that the informed manager can get a better information than the market, but with some noise, potentially at some cost.

Now, this noise, which we can consider as a measurement error, should be independent from all other random factors, and its amplitude should not depend on when the measurements occur. Also, for the sake of simplicity and since we already made this assumption, we keep on dealing with a risk-neutral trader. Or, another justification to just taking the expectation could be that these measurement errors are entirely diversifiable in the market.

So the simplest approach is to assume that the observed variable is read each time with a lognormal error, or the logarithm of the variable is read with a Gaussian error. We write ε the variance of this Gaussian variable. So in fact when the decision is made to invest as the noisy observed variable hits a level, the consequences of this decision will not be entirely deterministic but will depend on the result of the draw of this Gaussian variable U_i. We want the noise to be unbiased, so the mean of the variables has to be $-\frac{\varepsilon^2}{2}$.

So the total additional value generated by informed trading can be written

$$\mathbb{E}_{b'}\left[\sum_i \left(e^{-rT_d^i}G_d^i + e^{-rT_u^i}G_u^i\right)\right]$$

with

$$\begin{aligned}
\text{gain} &= G_d^i = M_d \frac{\Delta\left(X_t^\kappa - S_t^\kappa e^{\frac{M_d}{\phi}} e^{\kappa U_i}\right)}{\Delta X_t^\kappa} \\
&= \phi \ln\left(\frac{l_d}{h_d}\right)\left(1 - l_d e^{\kappa U_i}\right)
\end{aligned}$$

and when selling

$$\text{gain} = G_u^i = \phi \ln\left(\frac{h_u}{l_u}\right)\left(l_u e^{\kappa U_i} - 1\right).$$

One can see easily that if the profit function is such that $\kappa = 1$, then the noise does not change the value to the trader nor the optimal strategy. If the profit function is of a Cobb-Douglas type, the noise affects the value of the gain, and therefore the optimal strategy. In expectation, we obtain the following for the gains

$$\begin{aligned}
G_d^i &= \phi \ln\left(\frac{l_d}{h_d}\right)\left(1 - l_d e^{\frac{\varepsilon^2\left(\kappa-\kappa^2\right)}{2}}\right) \\
G_u^i &= \phi \ln\left(\frac{h_u}{l_u}\right)\left(l_u e^{\frac{\varepsilon^2\left(\kappa-\kappa^2\right)}{2}} - 1\right).
\end{aligned}$$

So, if κ is smaller than 1, the correction term is positive, and the expected utility is greater. In the limit case when minimum trading volume goes to zero, the total value to the informed agent is different if there is noise, but the optimal strategy will not be different. Indeed, the value function is the same, up to a multiplicative factor. The influence of a noisy information is therefore quite limited in our setting. By comparing the above equations with Proposition 38, we notice it corresponds to altering the market depth parameter ϕ.

The optimal strategy

The maximum value for the manager at the optimum is found by maximization with respect to the intervention levels and target zones. This gives the strategy followed by the manager at the optimum, as well as the additional yield he will obtain by entering such trades. We consider Platen and Schweizer's model, as the approach developed by Frey and Stremme unfortunately prevents finding easily the solution. Though, as their model is a more realistic approach, since it takes into account the size of the market, we will use it to find a realistic parameter for the depth of the market ϕ. By directly applying Proposition 38, we get

Proposition 39 *For the manager, the value of performing informed trades is*

$$\frac{4\phi\ln(k)\left(\frac{x}{k}-1\right)\left(\left(\frac{x}{k}\right)^{v}-\left(\frac{x}{k}\right)^{-v}+(k)^{v}-(k)^{-v}\right)}{\left((x)^{v}-(x)^{-v}\right)\left((x)^{2v}-(x)^{-2v}-\left(\frac{x}{k}\right)^{v}+\left(\frac{x}{k}\right)^{-v}-(k)^{v}+(k)^{-v}\right)}$$

where x is the width of the intervention band, and k is the relative impact on prices of the maximal transaction. Other parameters are summarized in $v=\frac{\sqrt{r}}{\kappa\sigma\sqrt{1-\rho}}$. Maximizing with respect to x gives the size of the optimal width for the intervention band. For very small transactions, ie if k goes towards zero, the gain at the limit is zero for the manipulator, but the intervention band is not reduced to zero and equals

$$\arg\max_{x>1}\frac{(x-1)\left(x^{v}-x^{-v}\right)}{\left(x^{v}-x^{-v}\right)\left(x^{2v}-x^{-2v}-x^{v}+x^{-v}\right)}$$

We illustrate this result with a numerical example. Let us consider, for example, a small company with a market capitalization of \$20 million. There are 1,000,000 shares, each worth \$20. To assess the depth of the market, we assume the following. If a manipulator who does not own any of these shares, wanted the price to increase by 10%, then a number of shares x would have to be traded such that $1.1=\frac{1{,}000{,}000}{1{,}000{,}000-x}$, that is trade around 91,000 shares, following the second market model we presented. Therefore, if we assume the same kind of relationship still holds for Follmer and Schweizer's model, we would expect to have ϕ solving $1.1=e^{\frac{91{,}000}{\phi}}$, that is $\phi=955,000$. Notice that if we assume the trader already holds a non negligible part of the market capitalization, the effect of transactions is even stronger on prices. Also, ϕ can be taken to be almost equal to the total number of shares on the market.

Let us assume in addition the volatility σ equals 40% (the volatility has to be rather high, since we consider a small capitalization), the interest rate r is 10%, and the correlation factor ρ 0.5. In these conditions, we get $v=1.12$.

We consider now that there is a maximal volume volume m. This maximal transaction size puts a constraint on the barriers l_u and l_d by expressing them as a function of h_u and h_d.

If we assume a symmetrical situation where $h_u=\frac{1}{h_d}$ and $l_u=\frac{1}{l_d}$, we obtain the following value, using Proposition 39

$$\frac{2\phi\ln\left(\frac{h_u}{l_u}\right)(l_u-1)\left((h_ul_u)^{v}-(h_ul_u)^{-v}+\left(\frac{h_u}{l_u}\right)^{v}-\left(\frac{h_u}{l_u}\right)^{-v}\right)}{(h_u)^{2v}-(h_u)^{-2v}-(h_ul_u)^{v}+(h_ul_u)^{-v}-\left(\frac{h_u}{l_u}\right)^{v}+\left(\frac{h_u}{l_u}\right)^{-v}}\frac{\sinh\left(\ln\left((h_u)^{\frac{\sqrt{r}}{\kappa\sigma\sqrt{1-\rho}}}\right)\right)}{\sinh\left(\ln\left((h_u)^{\frac{2\sqrt{r}}{\kappa\sigma\sqrt{1-\rho}}}\right)\right)}$$

with $v=\frac{\sqrt{r}}{\kappa\sigma\sqrt{1-\rho}}$ for the gain of future transactions. This can be simplified into

$$\frac{4\phi\ln\left(\frac{h_u}{l_u}\right)(l_u-1)\left((h_ul_u)^{v}-(h_ul_u)^{-v}+\left(\frac{h_u}{l_u}\right)^{v}-\left(\frac{h_u}{l_u}\right)^{-v}\right)}{\left((h_u)^{v}-(h_u)^{-v}\right)\left((h_u)^{2v}-(h_u)^{-2v}-(h_ul_u)^{v}+(h_ul_u)^{-v}-\left(\frac{h_u}{l_u}\right)^{v}+\left(\frac{h_u}{l_u}\right)^{-v}\right)}.$$

If we suppose there is a maximum of 2000 shares to trade it implies that $m=2000=\phi\ln\left(\frac{h_u}{l_u}\right)$ and therefore $\frac{h_u}{l_u}=e^{\frac{2}{955}}=1.0022$. So the size of the jumps

x	$k = 1.001$	$k = 1.01$	$k = 1.05$
1.1	2.0531×10^{-3}	$.01842$	4.8619×10^{-2}
1.2	2.0974×10^{-3}	1.9728×10^{-2}	7.3115×10^{-2}
1.3	2.1037×10^{-3}	2.0089×10^{-2}	8.1062×10^{-2}
1.4	2.085×10^{-3}	2.0053×10^{-2}	8.4025×10^{-2}
1.5	2.0492×10^{-3}	$.01979$	8.4671×10^{-2}
1.6	2.0022×10^{-3}	1.9387×10^{-2}	8.4037×10^{-2}
2.0	1.7724×10^{-3}	1.7245×10^{-2}	7.6525×10^{-2}
2.5	1.4995×10^{-3}	1.4622×10^{-2}	6.5533×10^{-2}
3.0	1.2822×10^{-3}	1.2515×10^{-2}	5.6337×10^{-2}

Table 8.1 Value of Manipulation vs. Maximal Trade Size

x	$v = 0.8$	$v = 1$	$v = 1.2$
1.1	2.3121×10^{-2}	$.01842$	1.5272×10^{-2}
1.2	2.5015×10^{-2}	1.9728×10^{-2}	1.6158×10^{-2}
1.3	$.02584$	2.0089×10^{-2}	1.6175×10^{-2}
1.4	2.6237×10^{-2}	2.0053×10^{-2}	1.5829×10^{-2}
1.5	2.6379×10^{-2}	$.01979$	1.5293×10^{-2}
1.6	$.02635$	1.9387×10^{-2}	$.01466$
2.0	2.5324×10^{-2}	1.7245×10^{-2}	1.2004×10^{-2}
2.5	2.3377×10^{-2}	1.4622×10^{-2}	9.3363×10^{-3}
3.0	$.02147$	1.2515×10^{-2}	7.4538×10^{-3}

Table 8.2 Influence of v on Manipulation Value

induced by the minimal transactions is around 0.2%. We can consider that such a size is reasonable so that these jumps are not noticed by the other agents in the market.

If we write $\frac{h_u}{l_u} = k$ and $h_u = x$, then the expression of the total gain can be written

$$W(x,v,k) = \frac{4\ln(k)\left(\frac{x}{k}-1\right)\left(\left(\frac{x}{k}\right)^{v}-\left(\frac{x}{k}\right)^{-v}+(k)^{v}-(k)^{-v}\right)}{\left((x)^{v}-(x)^{-v}\right)\left((x)^{2v}-(x)^{-2v}-\left(\frac{x}{k}\right)^{v}+\left(\frac{x}{k}\right)^{-v}-(k)^{v}+(k)^{-v}\right)}$$

See Table 8.1 on p. 154 for the total value to the manipulator for different minimal trading amounts (in units to be multiplied by ϕ). The maximum gain realized is clearly a function of the maximum trade size.

If the maximal size of a transaction goes to zero, then the profit realized by the manipulator goes also to zero. However, the optimal trading range converges to the maximum of

$$\frac{(x-1)\left(x^{v}-x^{-v}\right)}{\left(x^{v}-x^{-v}\right)\left(x^{2v}-x^{-2v}-x^{v}+x^{-v}\right)}$$

If $v = 1$ we find $x = 2^{\frac{1}{3}}$, that is $v \simeq 1.26$.

It appears therefore that setting a low maximal trade volume on the exchange so as to deter optimal manipulation trades would affect the total profit realized by the manager in a noticeable manner, therefore finding a justification. The influence of the parameter v can also be calculated (see Table 8.2 on p. 154).

The higher the exponent v, the lower the value of manipulation. The perpetual value generated by these manipulations in our simulation with $k = 1.05$ and $v = 1$ is slightly less than 10% of the company market capitalization.

The impact of informed trading on the evolution of prices

One can argue that the opportunistic trading is useful, as it keeps the market value of the company in line with the real value of its future cashflows. If we are in the case of using costly information, the price of this information should reflect precisely the additional yield due to informed trading. So this general cost can be thought of as the cot of being insured that market prices will not wander too far off real prices.

The distribution of controlled prices

The price process being controlled in a tunnel around the real price, its distribution will converge towards a stationary distribution. It is well known, for example, that the distribution of a Brownian Motion reflected in a tunnel converges towards an exponentially-shaped distribution (a uniform distribution if the Brownian Motion is driftless). A simple proof can be found in Dixit and Pindyck (1994). To model the distribution of the controlled price without assuming the stationary distribution has been reached would require to use proposition 15 and invert the Laplace transform.

Let us look at the singular control case: the term distribution of the price is that of the real value perturbed by the control, which we assume stationary, considering that managers have been intervening on this share for a long time.

The distribution of the yield is obtained by deriving the law of the sum of the Gaussian original return and the uniform distribution of the ergodic effect of the manipulation. We obtain

$$\mathbb{P}\left(\ln\left(\frac{S_t}{S_0}\right) \in dv\right) = dv \int_0^1 \frac{du}{\sqrt{2\pi t\sigma^2}} e^{-\frac{\left(v-(b-a)u-\left(\mu-\frac{\sigma^2}{2}\right)t-a\right)^2}{2t\sigma^2}}$$

with

$$a = \frac{\ln(h_d)}{\kappa\sigma\sqrt{2(1-\rho)}} \text{ and } b = \frac{\ln(h_u)}{\kappa\sigma\sqrt{2(1-\rho)}}.$$

This distribution would be, without any intervention from the informed trader:

$$\mathbb{P}\left(\frac{S_t}{S_0} \in dv\right) = \frac{dv}{\sqrt{2\pi t\sigma^2}} e^{-\frac{\left(v-\left(\mu-\frac{\sigma^2}{2}\right)t\right)^2}{2t\sigma^2}}.$$

Table 8.3 on p. 156 shows a comparison of returns actual volatility with or without manipulation. We compare

$$\int_{-\infty}^{+\infty} dv v^2 \int_0^1 \frac{du}{\sqrt{2\pi t\sigma^2}} e^{-\frac{\left(v-(b-a)u-\left(\mu-\frac{\sigma^2}{2}\right)t-a\right)^2}{2t\sigma^2}}$$

and

$$\int_{-\infty}^{+\infty} dv v^2 \frac{e^{-\frac{\left(v-\left(\mu-\frac{\sigma^2}{2}\right)t\right)^2}{2t\sigma^2}}}{\sqrt{2\pi t\sigma^2}} = t\sigma^2.$$

$a=-b$	Variance of manipulated returns
0.005	.16091
0.05	.16173
0.1	.1642
0.2	.1741
0.3	.1906
0.4	.2137
0.5	.2434
0.75	.34653
1.0	.4909
1.5	.92748
2.0	1.4809

Table 8.3 Variance of Manipulated Returns

The values of the parameters are the same as in the example discussed in the preceding section, and we look at a horizon of 1 year. The results are to be compared with the variance of non manipulated returns of 0.16 in our example.

Maintaining the share price in a tunnel around the real value changes the shape of the distribution of returns to market participants. The returns are more volatile. This added volatility is in fact the cost for the other agents of the manager's trading strategy.

The limit distribution will depend on two kind of factors: the ones related to the market itself (volatility, depth) and the ones related to the manipulator (costs, quality of the information). The correlation between the real value and the market value can be considered to belong to either one of these categories, depending if it is controlled by the information expenses of the manipulator, or if it is simply a fact that characterize the market for the stock under scrutiny.

The depth of the market, as measured by the coefficient ϕ has a very identifiable effect on the gains realized by the informed trader. Indeed, it is a multiplicative coefficient, and therefore only impacts the total gain, without affecting the optimal strategy. A deeper market means that the manager can perform more trades without influencing the market price adversely.

Increases in correlation between the real value and the market value increase the v coefficient. This is also the effect of increasing r, reducing σ or reducing κ. We can see easily that increases in v at the same time reduce the present value of the trades and the size of the trading range. So, as could be intuitively expected, correlation increases reduce the gain to the informed trader. A higher volatility on the contrary improves the gains from informed trading.

Why prevent informed trading?

We have seen that informed trading prevents market prices from being too far away from the real value. We asserted that it is a desirable objective, as the intuition suggests, but it is better to quantify this objective. Also, we have seen that the control by the informed trader induces a modification of the shape of returns' distributions and increases volatility. The impact of such an effect on stockholder's utilities should be equally quantified.

From the viewpoint of marked participants, the optimal manipulation, if any, should maximize a function expressing their interest in representative prices minus the cost of increased volatility. This could be expressed as

$$\sup_{h_u,h_d}\left\{\mathbb{E}\left[\int_0^\infty dte^{-\rho t}u\left(\frac{S_t}{X_t}\right)\right]-a\left(\frac{R_{(h_u,h_d)}}{R_{(\infty,0)}}\right)\right\}$$

where u is a utility function representing the aggregate preferences of market participants as for the discrepancy between real and market value, ρ is an intertemporal preferences coefficient, and $R_{(h_u,h_d)}$ is a measure of the risk of returns if there is manipulation. a measures the cost of the additional risk relative to the no-manipulation case. The expression $\mathbb{E}\left[\int_0^\infty dte^{-\rho t}u\left(\frac{S_t}{X_t}\right)\right]$ is known explicitly thanks to Proposition 36, and the risk measure R will be expressed as a function of the distribution of manipulated market prices which have been derived explicitly earlier.

This optimization program yields the level of manipulation at which market participants' interests are the highest. The regulation of the exchange will necessarily try to attain this optimum, and therefore should try to discourage informed traders from setting their intervention levels at different levels from the ones determined by the program. The tools they can use are minimal or maximal trading volumes, fixed costs, or finally the fine or punishment incurred by insider traders. It may be optimal for market participants that no manipulation be performed on the stock price, in which case punishments would be very severe and minimal volumes high, but it can be the case that realigning prices counterbalances the cost bearing with manipulations.

Concluding remarks

We have analyzed how an informed agent can maximize its wealth by trading opportunistically the shares of a company, based on its knowledge of the real value of the future cashflows this equity will generate. By buying at low prices and selling at high prices relative to the real value, the agent impacts the market price and prevents the market value from wandering too far away from the real value. We have derived the profits generated by such a strategy, and some properties of the controlled market price.

Such a situation arises in particular when a firm manager trades his firm's stock, which has been shown to be one of the primary sources of income for managers. We have quantified a relationship between some market properties, some parameters linked to the manager, and the long-term distribution of share prices. When managers perform trades, they tend to increase the volatility of share prices, but maintain this price within a range around the real value of these shares.

The gains generated by the informed trader can be thought of as the cost for the rest of the market of being guaranteed that prices reflect somewhat the value of the company.

The empirical analysis of both models we proposed to explain the influence of volume on prices is left for further research. Also, testing the relationship between trades induced by companies managers and the long-term distribution of shares performance is an empirical study that is still to be done.

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Appendix

Platen and Schweizer Theorem

We provide here a simplified proof of Platen and Schweizer's theorem mentioned in the second section.

An application of Itô's theorem to 8.2 gives

$$0 = \gamma dK + \zeta^{\cdot} dt + \zeta' dK + \frac{1}{2}\zeta'' d[K] + dU$$

with $K = \ln(S)$. By Meyer's identification theorem, we find that

$$\begin{aligned} 0 &= \gamma dM + \zeta' dM + dB \\ 0 &= \gamma dA + \zeta^{\cdot} dt + \zeta' dA + \frac{1}{2}\zeta'' d[K] + dN \end{aligned}$$

where $K = A + M$ and $U = N + B$ are the semi-martingale decomposition of the two processes. We obtain

$$\begin{aligned} dM &= -\frac{dB}{\gamma + \zeta'} \text{ so } d[K] = d[M] = \frac{d[U]}{(\gamma + \zeta')^2} \text{ and} \\ 0 &= \gamma dK + \zeta^{\cdot} dt + \zeta' dK + \frac{1}{2}\zeta'' \frac{d[U]}{(\gamma + \zeta')^2} + dU \end{aligned}$$

from which we get

$$dK = \frac{-\zeta^{\cdot} dt - dU}{(\gamma + \zeta')} - \frac{1}{2}\zeta'' \frac{d[U]}{(\gamma + \zeta')^2}.$$

Now, we can write by applying Itô to K

$$\begin{aligned} dK &= \frac{dS}{S} - \frac{1}{2S^2} d[S], \text{ so} \\ \frac{dS}{S} &= \frac{-\zeta^{\cdot} dt - dU}{(\gamma + \zeta')} - \frac{1}{2}\zeta'' \frac{d[U]}{(\gamma + \zeta')^3} + \frac{1}{2S^2} d[S] \end{aligned}$$

and this allows us to write that $d[S] = \frac{S^2}{(\gamma + \zeta')^2} d[U]$ so finally

$$\frac{dS}{S} = \frac{-\zeta^{\cdot} dt - dU}{(\gamma + \zeta')} - \frac{1}{2} \frac{(\zeta'' + \gamma + \zeta')\, d[U]}{(\gamma + \zeta')^3}$$

and it completes the proof.

The case of a singular control

In this subsection we study the law of the Brownian Motion, and its cost of control, when the control is applied in a singular way, that is an infinite number of times and of an infinitesimal size. In his book, Harrison (1985) gives a very good overview of this problem, following an analytical method. Our approach here will be more probabilistic. Note that Harrison also gives the differential equation solved by the expectation $\mathbb{E}_x \left[\int_0^\infty dt e^{-rt} f\left(\widehat{B}_t\right) \right]$ in the singular case.

This problem corresponds to the preceding one with the size of the control jumps going down to zero. For our applications, it is directly possible to obtain this limit. This subsection only illustrates some connections with the concept of local time. An introduction to the definition and properties of local times for continuous semi-martingales can be found in Yor (1997), or Revuz and Yor (1991).

The controlled process, at the limit, is a reflected Brownian Motion in a tunnel. Every time the process hits one of the two barriers, it receives an infinitesimal impulsion until it bounces back towards the center of the tunnel. The limit of the size of such impulsions times their number is an intuitive description of the local time.

Proposition 40 *The present value of the cost of controlling the process is given by*

$$\begin{aligned} C(x) &= \frac{1}{2(b-a)r} \\ &\quad \left(\left(C_a + \frac{C_b}{\cosh\left((b-a)\sqrt{2r}\right)}\right)\frac{\sinh\left((x-a)\sqrt{2r}\right)}{\sinh\left((b-a)\sqrt{2r}\right)}\right. \\ &\quad \left. + \left(C_b + \frac{C_a}{\cosh\left((b-a)\sqrt{2r}\right)}\right)\frac{\sinh\left((b-x)\sqrt{2r}\right)}{\sinh\left((b-a)\sqrt{2r}\right)}\right) \end{aligned}$$

where C_a and C_b are the proportional costs prevailing at the barriers a and b.

Proof. We first of all will need the following result which can be found in Borodin and Salminen (1996).

Lemma 41 *Using the fact that $T_x(|B|) = T_x(B) \wedge T_{-x}(B)$ it is easy to show that*

$$\mathbb{E}\left[e^{-rT_x(|B|)}\right] = \frac{1}{\cosh\left(x\sqrt{2r}\right)}.$$

We will consider the Brownian Motion reflected in the tunnel defined by a and b. The cost of intervention will have to be determined as the limit of the cost in the impulse control case when the size of the jumps go to zero. If we write $C(\varepsilon)$ for the cost, with $\varepsilon = b - b'$ or $a - a'$, we would write $C = \lim_{\varepsilon\to 0} \frac{C(\varepsilon)-c}{\varepsilon}$ where c is any fixed component of the cost. It is clear that we must have $c = 0$ for the proportional cost not to explode.

Now, we consider the controlled process from the barrier b up to its next hitting time of a, and therefore the next intervention. The total cost of intervention up to that time is given by

$$C_b c_\varepsilon^b = C_b \sum_n \mathbb{I}_{T_b \le T_b^n \le T_a} e^{-rT_b^n} \varepsilon$$

where the T_b^n are the successive instants when the process is pushed from b to b' (by a distance ε). c_ε^a is defined symetrically.

Using the same approach as in the preceding proofs, we write the expectation of the costs starting from a as a function of the costs starting from b and reciprocally:

$$\begin{cases} C(a) = \mathbb{E}_{a+\varepsilon}\left[C_a c_\varepsilon^a + e^{-rT_b} C(b)\right] \\ C(b) = \mathbb{E}_{b-\varepsilon}\left[C_b c_\varepsilon^b + e^{-rT_a} C(a)\right] \end{cases}$$

Solving the limit of this system as ε goes to zero will give the cost of the singular control. First of all, we know that due to the symetrical situation we have

$\mathbb{E}_{a+\varepsilon}\left[c_\varepsilon^a\right] = \mathbb{E}_{b-\varepsilon}\left[c_\varepsilon^b\right]$. More precisely,

$$\begin{aligned}
p &= \lim_{\varepsilon\to 0} \mathbb{E}_{b-\varepsilon}\left[c_\varepsilon^b\right] \\
&= \lim_{\varepsilon\to 0} \varepsilon \sum_{k=1}^{\infty} k \mathbb{E}_\varepsilon \left[e^{-rT_0} \mathbb{I}_{T_0 \le T_{b-a}}\right]^k \qquad \mathbb{P}_\varepsilon\left(T_{b-a} \le T_0\right) \\
&= \lim_{\varepsilon\to 0} \varepsilon \sum_{k=1}^{\infty} k \left(\frac{\sinh\left((b-a-\varepsilon)\sqrt{2r}\right)}{\sinh\left((b-a)\sqrt{2r}\right)}\right)^k \frac{\varepsilon}{b-a} \\
&= \frac{\sinh^2\left((b-a)\sqrt{2r}\right)}{2(b-a)r\cosh^2\left((b-a)\sqrt{2r}\right)}.
\end{aligned}$$

In addition to that, by symetry we also have $\mathbb{E}_{a+\varepsilon}\left[e^{-rT_b}\right] = \mathbb{E}_{b-\varepsilon}\left[e^{-rT_a}\right]$. At the limit, the controlled process converges to a reflected Brownian Motion between the barriers, so the Laplace transform of the first hitting time of the opposite barrier should be given by the lemma:

$$q = \lim_{\varepsilon\to 0} \mathbb{E}_{b-\varepsilon}\left[e^{-rT_a}\right] = \mathbb{E}\left[e^{-rT_{b-a}(|B|)}\right] = \frac{1}{\cosh\left((b-a)\sqrt{2r}\right)}.$$

This can also be shown by simply expressing $\mathbb{E}_{b-\varepsilon}\left[e^{-rT_a}\right]$ and taking the limit.

Solving the system gives

$$\begin{aligned}
C(a) &= C_a p + q\left(C_b p + qC(a)\right) \\
&= \frac{p}{1-q^2}\left(C_a + qC_b\right) \\
&= \left(C_a + \frac{C_b}{\cosh\left((b-a)\sqrt{2r}\right)}\right) \frac{\sinh^2\left((b-a)\sqrt{2r}\right)}{2(b-a)r\left(\cosh^2\left((b-a)\sqrt{2r}\right)-1\right)} \\
&= \frac{C_a + \frac{C_b}{\cosh\left((b-a)\sqrt{2r}\right)}}{2(b-a)r}.
\end{aligned}$$

Starting from x, the cost $C(x)$ is equal to

$$\begin{aligned}
& \mathbb{E}_x\left[e^{-rT_a}\mathbb{I}_{T_a \le T_b}\right] C(a) + \mathbb{E}_x\left[e^{-rT_b}\mathbb{I}_{T_b \le T_a}\right] C(b) \\
= & \frac{\sinh\left((x-a)\sqrt{2r}\right)}{\sinh\left((b-a)\sqrt{2r}\right)} C(a) + \frac{\sinh\left((b-x)\sqrt{2r}\right)}{\sinh\left((b-a)\sqrt{2r}\right)} C(b) \\
= & \frac{1}{2(b-a)r} \\
& \left(\left(C_a + \frac{C_b}{\cosh\left((b-a)\sqrt{2r}\right)}\right) \frac{\sinh\left((x-a)\sqrt{2r}\right)}{\sinh\left((b-a)\sqrt{2r}\right)} \right. \\
& \left. + \left(C_b + \frac{C_a}{\cosh\left((b-a)\sqrt{2r}\right)}\right) \frac{\sinh\left((b-x)\sqrt{2r}\right)}{\sinh\left((b-a)\sqrt{2r}\right)}\right)
\end{aligned}$$

and this ends the proof. ■

Notice that if $x = 0$ and $b = -a$ the expression simplifies to

$$\begin{aligned}
C(0) &= \frac{1}{4br} \frac{\sinh\left(b\sqrt{2r}\right)}{\sinh\left(2b\sqrt{2r}\right)} \\
& \qquad \left(C_a + C_b\right)\left(1 + \frac{1}{\cosh\left(2b\sqrt{2r}\right)}\right).
\end{aligned}$$

Chapter 9 CONCLUSION

The objective of this work was to show that real options have a lot to benefit from exotic options analogies and probabilistic methods. We have focused on a few specific examples, where we insisted on the relationship between real and exotic options.

Focusing on the specific issue of delays in investment decisions, we have used a probabilistic method inspired from exotic options to obtain closed-form formulae for the valuation of a real option. This has allowed us to gain insights on the optimal decision rules under this particular constraint. We have provided a general valuation framework for investment opportunities relying on the computation of first passage times. We showed that the delay existing between the investment decision and its implementation has important valuation consequences. The approach we presented in the second chapter is general enough to allow us to extend the scope of our analysis to other rigidities in the investment process. We focused on the delay existing between the investment decision and its real implementation.

We have also seen how large entities and smaller entities face different constraints when they contemplate investing in a project. In a competition situation where the first to invest totally preempts the project, we have exposed how a particular class of options, Parisian American options, allows to model the combined constraints faced by investors. The largest firm has an option to invest which can be exercised only under a given barrier (over which the smaller firms invests immediately) and according to its investment delay constraints. We have given a new result pertaining to the first instant a Brownian Motion hits a level or spends more than a given amount of time above a lower level. This result allowed us to derive a pricing formula for these options. From the technical viewpoint, the same approach helps value options where functionals of excursions intervene (such as the first instant when the area of an excursion reaches a certain level).

The simple model we developed in Chapter 5 shows that the noise in the information available to investors can explain part of their investment behavior. This model relies essentially on probabilistic techniques and concepts, and departs from the traditional approach in that respect.

In Chapter 6, we focused on how to hedge an option with other options, when there are transaction costs. Such an issue arises if a corporation that consumes or produces traded commodities needs to hedge its real options. The most appropriate hedging tools could be traded financial options on the commodities in question. We have proposed a proof of the convergence of Leland's scheme towards a non-linear PDE in a general setting. As the setting includes transaction costs, an allocation strategy minimizing these costs can be followed. It has clearly appeared there is a possibility to benefit from lower transactions fees by thoroughly choosing how to hedge a derivative. But the optimal strategy and the price of the hedged derivative are solutions of complicated equations, which we were only able to approach in simple cases. For example, if we consider usual path-dependent options, like barrier or lookback options, and contemplate hedging them with plain-vanilla options while minimizing cumulated transaction costs, the equation has to be solved numerically.

We have proposed a new class of barrier derivatives in Chapter 7, Switch options, that allows to mitigate the losses due to the "knock-out" effect of classical barrier options. These derivative products also constitute a hedging tool of the business risk linked to entry or exit decision, as they allow to replicate or complement the payoffs of real options. To hedge this risk with Switch options, it is required that the underlying business variable be traded, which restraints the use of these derivatives mostly to commodities firms. As a tool to price Switch options, we have derived the joint law of the Brownian Meander and its running maximum.

We have finally analyzed in Chapter 8 how an informed agent can maximize its wealth by trading opportunistically the shares of a company, based on its knowledge of the real value of the future cashflows this equity will generate. By buying at low prices and selling at high prices relative to the real value, the agent impacts the market price and prevents the market value from wandering too far away from the real value. We have derived the profits generated by such a strategy, and some properties of the controlled market price. We carried out this analysis from the perspective of real option analysis, the decisions to buy or sell stock being similar to entry or exit decisions. We have quantified a relationship between some market properties, some parameters linked to the manager, and the long-term distribution of share prices. When managers perform trades, they tend to increase the volatility of share prices, but maintain this price within a range around the real value of these shares. The gains generated by the informed trader can be thought of as the cost for the rest of the market of being guaranteed that prices reflect somewhat the value of the company.